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A Gravity Theory on Noncommutative Spaces

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The concepts necessary for an algebraic construction of a gravity theory on noncommutative spaces are introduced. The θ-deformed diffeomorphisms are studied and a tensor calculus is defined. This leads to a deformed Einstein-Hilbert action which is invariant with respect to deformed diffeomorphisms.

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1. Noncommutative Spaces

It is expected that in order to obtain a better understanding of physics at short distances and in order to cure the problems occuring when trying to quantize gravity one has to change the nature of space-time in a fundamental way. One way to do so is to implement noncommutativity by taking coordinates which satisfy the commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = C^{\mu\nu}(\hat{x}) \neq 0\,. \tag{1.1}$$

The function $C^{\mu\nu}(\hat{x})$ is unknown. For physical reasons it should be a function that vanishes at large distances where we experience the commutative world and may be determined by experiments [1]. We denote the algebra generated by noncommutative coordinates $\hat{x}^\mu$ which are subject to the relations (1.1) by $\hat{\mathscr{A}}$ (*algebra of noncommutative functions*). In what follows we will exclusively consider the θ-deformed case which may at very short distances provide a reasonable approximation for $C^{\mu\nu}(\hat{x})$

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} = \text{const.} \tag{1.2}$$

but we note that the algebraic construction presented here can be generalized to more complicated noncommutative structures of the above type which possess the Poincaré-Birkhoff-Witt (PBW) property.

2. Symmetries on Deformed Spaces

In general the commutation relations (1.1) are not covariant with respect to undeformed symmetries. For example the canonical commutation relations (1.2) break Lorentz symmetry if we assume that the noncommutativity parameters $\theta^{\mu\nu}$ do not transform.

The question arises whether we can *deform* the symmetry in such a way that it acts consistently on the deformed space (i.e. leaves the deformed space invariant) and such that it reduces to the undeformed symmetry in the commutative limit. The answer is yes: Lie algebras can be deformed in the category of Hopf algebras (Hopf algebras coming from a Lie algebra are also called Quantum Groups)[1]. Quantum group symmetries lead to new features of field theories on noncommutative spaces. Because of its simplicity, θ-deformed spaces are very well-suited to study those.

In the following we will construct explicitly a θ-deformed version of diffeomorphisms which consistently act on the noncommutative space (1.2). It is possible to construct a gravity theory which is invariant with respect to these deformed diffeomorphisms [2, 3, 4].

3. Diffeomorphisms

Diffeomorphisms are generated by vector-fields ξ. Acting on functions, vector-fields are represented as linear differential operators $\xi = \xi^\mu \partial_\mu$. Vector-fields form a Lie algebra Ξ with the Lie bracket given by

$$[\xi, \eta] = \xi \times \eta$$

[1]To be more precise the universal enveloping algebra of a Lie algebra can be deformed. The universal enveloping algebra of any Lie algebra is a Hopf algebra and this gives rise to deformations in the category of Hopf algebras.

where $\xi \times \eta$ is defined by its action on functions

$$(\xi \times \eta)(f) = (\xi^\mu(\partial_\mu \eta^\nu)\partial_\nu - \eta^\mu(\partial_\mu \xi^\nu)\partial_\nu)(f).$$

The Lie algebra of *infinitesimal diffeomorphisms* Ξ can be embedded into its universal enveloping algebra which we want to denote by $\mathscr{U}(\Xi)$. The universal enveloping algebra is an associative algebra and possesses a natural Hopf algebra structure. The coproduct is defined as follows on the generators[2]:

$$\begin{aligned} \Delta : \mathscr{U}(\Xi) &\rightarrow \mathscr{U}(\Xi) \otimes \mathscr{U}(\Xi) \\ \Xi \ni \xi &\mapsto \Delta(\xi) := \xi \otimes 1 + 1 \otimes \xi. \end{aligned} \tag{3.1}$$

For a precise definition and more details on Hopf algebras we refer the reader to text books [5]. For our purposes it shall be sufficient to note that the coproduct implements how the Hopf algebra acts on a product in a representation algebra (Leibniz-rule). Scalar fields are defined by their transformation property with respect to infinitesimal coordinate transformations:

$$\delta_\xi \phi = -\xi \phi = -\xi^\mu(\partial_\mu \phi). \tag{3.2}$$

The product of two scalar fields is transformed using the Leibniz-rule

$$\delta_\xi(\phi \psi) = (\delta_\xi \phi)\psi + \phi(\delta_\xi \psi) = -\xi^\mu(\partial_\mu \phi \psi) \tag{3.3}$$

such that the product of two scalar fields transforms again as a scalar.

Similarly one studies tensor representations of $\mathscr{U}(\Xi)$. For example vector fields are introduced by the transformation property

$$\begin{aligned} \delta_\xi V_\alpha &= -\xi^\mu(\partial_\mu V_\alpha) - (\partial_\alpha \xi^\mu) V_\mu \\ \delta_\xi V^\alpha &= -\xi^\mu(\partial_\mu V^\alpha) + (\partial_\mu \xi^\alpha) V^\mu. \end{aligned}$$

4. Deformed Diffeomorphisms

The concepts introduced in the previous subsection can be deformed in order to establish a consistent tensor calculus on the noncommutative space-time algebra (1.2). In this context it is necessary to account the full Hopf algebra structure of the universal enveloping algebra $\mathscr{U}(\Xi)$.

We want to deform the structure map (3.1) of the Hopf algebra $\mathscr{U}(\Xi)$ in such a way that the resulting deformed Hopf algebra which we denote by $\mathscr{U}(\hat{\Xi})$ consistently acts on $\hat{\mathscr{A}}$. Let $\mathscr{U}(\hat{\Xi})$ be generated as algebra by elements $\hat{\delta}_\xi$, $\xi \in \Xi$. We leave the algebra relation undeformed

$$[\hat{\delta}_\xi, \hat{\delta}_\eta] = \hat{\delta}_{\xi \times \eta} \tag{4.1}$$

but we deform the co-sector

$$\Delta \hat{\delta}_\xi = e^{-\frac{i}{2} h \theta^{\rho\sigma} \hat{\partial}_\rho \otimes \hat{\partial}_\sigma} (\hat{\delta}_\xi \otimes 1 + 1 \otimes \hat{\delta}_\xi) e^{\frac{i}{2} h \theta^{\rho\sigma} \hat{\partial}_\rho \otimes \hat{\partial}_\sigma}, \tag{4.2}$$

[2] The structure maps are defined on the generators $\xi \in \Xi$ and the universal property of the universal enveloping algebra $\mathscr{U}(\Xi)$ assures that they can be uniquely extended as algebra homomorphisms (respectively anti-algebra homomorphism in case of the antipode S) to the whole algebra $\mathscr{U}(\Xi)$.

where $[\hat{\partial}_\rho, \hat{\delta}_\xi] = \hat{\delta}_{(\partial_\rho \xi)}$. The deformed coproduct (4.2) reduces to the undeformed one (3.1) in the limit $\theta \to 0$. We need a differential operator acting on fields in $\hat{\mathscr{A}}$ which represents the algebra (4.1). Let us consider the differential operator

$$\hat{X}_\xi := \sum_{n=0}^{\infty} \frac{1}{n!}(-\frac{i}{2})^n \theta^{\rho_1\sigma_1} \cdots \theta^{\rho_n\sigma_n} (\hat{\partial}_{\rho_1} \cdots \hat{\partial}_{\rho_n} \xi^\mu) \hat{\partial}_\mu \hat{\partial}_{\sigma_1} \cdots \hat{\partial}_{\sigma_n}. \tag{4.3}$$

Then indeed we have

$$[\hat{X}_\xi, \hat{X}_\eta] = \hat{X}_{\xi \times \eta}. \tag{4.4}$$

It is therefore reasonable to introduce scalar fields $\hat{\phi} \in \hat{\mathscr{A}}$ by the transformation property

$$\hat{\delta}_\xi \hat{\phi} = -(\hat{X}_\xi \hat{\phi}).$$

Let us work out the action of the differential operators $\hat{X}_\xi$ on the product of two fields. A calculation [2] shows that

$$(\hat{X}_\xi(\hat{\phi}\hat{\psi})) = \mu \circ (e^{-\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho \otimes \hat{\partial}_\sigma} (\hat{X}_\xi \otimes 1 + 1 \otimes \hat{X}_\xi) e^{\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho \otimes \hat{\partial}_\sigma} \hat{\phi} \otimes \hat{\psi}).$$

This means that the differential operators $\hat{X}_\xi$ act via a *deformed Leibniz rule* on the product of two fields. Comparing with (4.2) we see that the deformed Leibniz rule of the differential operator $\hat{X}_\xi$ is exactly the one induced by the deformed coproduct (4.2):

$$\hat{\delta}_\xi(\hat{\phi}\hat{\psi}) = e^{-\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho \otimes \hat{\partial}_\sigma} (\hat{\delta}_\xi \otimes 1 + 1 \otimes \hat{\delta}_\xi) e^{\frac{i}{2}h\theta^{\rho\sigma}\hat{\partial}_\rho \otimes \hat{\partial}_\sigma} (\hat{\phi}\hat{\psi}) = -\hat{X}_\xi \triangleright (\hat{\phi}\hat{\psi}).$$

Hence, the deformed Hopf algebra $\mathscr{U}(\hat{\Xi})$ is indeed represented on scalar fields $\hat{\phi} \in \hat{\mathscr{A}}$ by the differential operator $\hat{X}_\xi$. The scalar fields form a $\mathscr{U}(\hat{\Xi})$-module algebra.

Up to now we have seen the following:

- Diffeomorphisms are generated by vector-fields $\xi \in \Xi$ and the universal enveloping algebra $\mathscr{U}(\Xi)$ of the Lie algebra Ξ of vector-fields possesses a natural Hopf algebra structure defined by (3.1).
- The algebra of scalar fields $\phi \in \mathscr{A}$ is a $\mathscr{U}(\Xi)$-module algebra.
- The universal enveloping algebra $\mathscr{U}(\Xi)$ can be deformed to a Hopf algebra $\mathscr{U}(\hat{\Xi})$ defined in (4.1,4.2).
- $\mathscr{U}(\hat{\Xi})$ consistently acts on the algebra of noncommutative functions $\hat{\mathscr{A}}$, i.e. the algebra of noncommutative functions is a $\mathscr{U}(\hat{\Xi})$-module algebra.
- Regarding $\mathscr{U}(\hat{\Xi})$ as the underlying "symmetry" of the gravity theory to be built on the noncommutative space $\hat{\mathscr{A}}$, we established a full tensor calculus as representations of the Hopf algebra $\mathscr{U}(\hat{\Xi})$.

5. Noncommutative Geometry

Based on deformed diffeomorphisms it is possible to introduce covariant derivatives, curvature and torsion and to define a metric [2, 3, 4]. This leads to a curvature scalar. Introducing in addition the star-determinant of the vierbein, one can construct a Einstein-Hilbert action which is invariant with respect to deformed diffeomorphisms. It is a deformation of the usual Einstein-Hilbert action. Using the star-product formalism it is possible to map the algebraic quantities to functions depending on commutative variables. Then it is possible to study explicitly deviations of the undeformed theory in orders of a deformation parameter [4, 2]. Very interesting is also to study a generalization of the above concepts to a more general class of noncommutative structures given by a twist [3]. This class contains in particular lattice-like spacetime algebras which may indeed provide a regularization of the field theory under consideration.

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