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$\Delta I = 3/2$ kaon weak matrix elements with non-zero total momentum

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We present preliminary results for the $\Delta I = 3/2$ kaon decay matrix elements using domain wall fermions and the DBW2 gauge action at one coarse lattice spacing corresponding to $a^{-1} = 1.3$ GeV. We employ an extention of the Lellouch and Lüscher formula for non-zero total momentum to extract the infinite volume, center-of-mass system decay amplitudes. We compare the result of ReA_2 with previous results calculated by several methods, and experiment. We also show the $I = 2$ $\pi\pi$ scattering phase shift.

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1. Introduction

Non-leptonic kaon decay includes interesting subjects such as, the $\Delta I = 1/2$ selection rule and the CP violation parameter ε'/ε. However, it is difficult to calculate the $K \to \pi\pi$ weak matrix element directly on lattice due to difficulty of calculation of the two-pion state in finite volume, which was pointed out by Maiani and Testa [1]. So far many groups employ an indirect method [2] using Chiral perturbation theory (ChPT) to avoid the difficulty. In the indirect method $K \to \pi\pi$ process is reduced to $K \to \pi$ and $K \to 0$ processes. Since in the decay process the final state interaction of the two-pion is expected to play an important role, the reduction may cause systematic errors.

For the direct method, where the two-pion state is calculated on lattice, we need complicated calculations and analyzes, *e.g.*, diagonalization of a matrix of correlation functions [3], to treat the two-pion state with non-zero relative momentum on lattice. However, recently Kim [4] reported an exploratory study of the direct calculation with H-parity boundary conditions, where complicated analyzes are not required. He succeeded to extract the two-pion state with non-zero relative momentum from the ground state contribution of correlation functions, because the two-pion state with zero momentum is prohibited by the boundary condition. In the non-zero total momentum (Lab) system, $|\vec{P}| \neq 0$, the ground state of the two-pions is $|\pi(0)\pi(\vec{P})\rangle$, which is related to the two-pion state with the non-zero relative momentum in center-of-mass (CM) system. Thus, we can extract the two-pion state with non-zero momentum from the ground state contributions as well as in the H-parity boundary case. However, we cannot naively apply the formula proposed by Lellouch and Lüscher (LL) [5] to the calculation of the Lab system, because the original formula is derived in the CM system. Before the simulation, we have to modify the LL formula to connect the finite volume decay amplitude in the Lab system to that of the CM system in infinite volume. Very recently, Kim *et al.* [6] and Christ *et al.* [7] suggested the formula which is an extension of the LL formula for the Lab system calculation. Here we attempt to apply this extended formula to the calculation of the $\Delta I = 3/2$ kaon weak matrix elements with domain wall fermions and the DBW2 gauge action at a single coarse lattice spacing.

2. Methods

Two groups [6, 7] proposed a formula between a decay amplitude in infinite $|A|$ (CM system) and finite volume $|M|$ (Lab system),

$$|A|^2 = 8\pi\gamma^2 \left(\frac{E_{\pi\pi}}{p}\right)^3 \left\{ p'\frac{\partial \delta}{\partial p'} + p'\frac{\partial \phi_{\vec{P}}}{\partial p'} \right\}_{p'=p} |M|^2, \tag{2.1}$$

where γ is a boost factor, $E_{\pi\pi}$ is the two-pion energy in the CM system, $p^2 = E_{\pi\pi}^2/4 - m_\pi^2$, and δ is the scattering phase shift of the final state interaction. The function $\phi_{\vec{P}}$ with $\vec{P}$ being the total momentum, derived by Rummukainen and Gottlieb [8], is defined by

$$\tan\phi_{\vec{P}}(q) = -\frac{\gamma q \pi^{3/2}}{Z_{00}^{\vec{P}}(1;q^2;\gamma)}, \tag{2.2}$$

where $q^2 = (pL/2\pi)^2$, and

$$Z_{00}^{\vec{P}}(1;q^2;\gamma) = \frac{1}{\sqrt{4\pi}} \sum_{\vec{n}\in\mathbb{Z}^3} \frac{1}{n_1^2 + n_2^2 + \gamma^{-2}(n_3+1/2)^2 - q^2}, \tag{2.3}$$

in the $\vec{P} = (0, 0, 2\pi/L)$ case. The formula eq. (2.1) is valid only for on-shell decay amplitude, *i.e.* $E_{\pi\pi} = m_K$ as is LL formula [5].

We calculate the four-point function for $\Delta I = 3/2$ $K \to \pi\pi$ decay with total momentum $\vec{P} = \vec{0}$ and $(0, 0, 2\pi/L)$. The four-point function $G_i(t)$ is defined by

$$G_i(t) = \langle 0|\pi^+\pi^-(\vec{P}, t_\pi) O_i^{3/2}(t) K^0(\vec{P}, t_K)|0\rangle, \tag{2.4}$$

where the operators $O_i^{3/2}$ are lattice operators entering $\Delta I = 3/2$ weak decays

$$O_{27,88}^{3/2} = (\bar{s}^a d^a)_L \left[(\bar{u}^b u^b)_{L,R} - (\bar{d}^b d^b)_{L,R} \right] + (\bar{s}^a u^a)_L (\bar{u}^b d^b)_{L,R} \tag{2.5}$$

$$O_{m88}^{3/2} = (\bar{s}^a d^b)_L \left[(\bar{u}^b u^a)_R - (\bar{d}^b d^a)_R \right] + (\bar{s}^a u^b)_L (\bar{u}^b d^a)_R \tag{2.6}$$

with $(\bar{q}q)_{L,R} = \bar{q}\gamma_\mu(1 \pm \gamma_5)q$ and a, b being color indices. $O_{27}^{3/2}$ and $O_{88}^{3/2}$ are the operators in the (27,1) and (8,8) representations of $SU(3)_L \otimes SU(3)_R$ with $I = 3/2$, respectively. $O_{m88}^{3/2}$ equals $O_{88}^{3/2}$ with its color summation changed to cross the two currents. We also calculate four-point function for two-pions and the two-point function of a kaon with zero and non-zero total momenta, to obtain those energies and amplitudes.

We employ the domain wall fermion action with the domain wall height $M = 1.8$, the fifth dimension length $L_s = 12$ and the DBW2 gauge action with $\beta = 0.87$ corresponding to $a^{-1} = 1.3$ GeV. The lattice size is $L^3 \cdot T = 16^3 \cdot 32$, where the spatial volume corresponds to about 2.4 fm. We use four light quark masses, $m_u = 0.015, 0.03, 0.04$ and 0.05, for the chiral extrapolation of the decay amplitudes, and six strange quark masses, $m_s = 0.12, 0.18, 0.24, 0.28, 0.35$ and 0.44, for the interpolation of the amplitudes to the on-shell point. We fix the two-pion operator at $t_\pi = 0$, while we employ three source points $t_K = 16, 20$ and 25 for the kaon operator to investigate the t_K dependence of the statistical error of the Lab system decay amplitude and to check the consistency of these results. The numbers of configuration used are 111 for $t_K = 16$ and 20, and 100 for $t_K = 25$. A quark propagator is calculated by averaging quark propagators with periodic and anti-periodic boundary conditions for the time direction to obtain a propagator with $2T$ periodicity. We also use a wall source with Coulomb gauge fixing for the quark propagators.

3. Results

In order to utilize the eq. (2.1) we need to measure derivatives of the scattering phase shift for the final state and evaluate the function $\phi_{\vec{P}}(q)$ defined in eq. (2.3). The phase shift $\delta(p)$ is evaluated from the two-pion energy by the finite volume formula [8, 9]. We carry out a global fitting of

$$T(m_\pi, p) = \frac{\tan\delta(p)}{p} \cdot \frac{E_{\pi\pi}}{2} \tag{3.1}$$

with a polynomial function $a_{10}m_\pi^2 + a_{20}m_\pi^4 + a_{01}p^2$ to obtain the derivative of δ with respect to p. The measured value of T (left graph) and δ (right graph), and their fitting results are plotted in Fig. 1. The prediction of δ from ChPT with experiment is also shown in the right graph.

To determine off-shell decay amplitudes in finite volume, we define ratio of correlation functions $R_i(t)$ for $i = 27, 88$ and $m88$ as

$$R_i(t) = \frac{\sqrt{3}G_i(t)Z_{\pi\pi}Z_K}{G_{\pi\pi}(t)G_K(t)}, \tag{3.2}$$

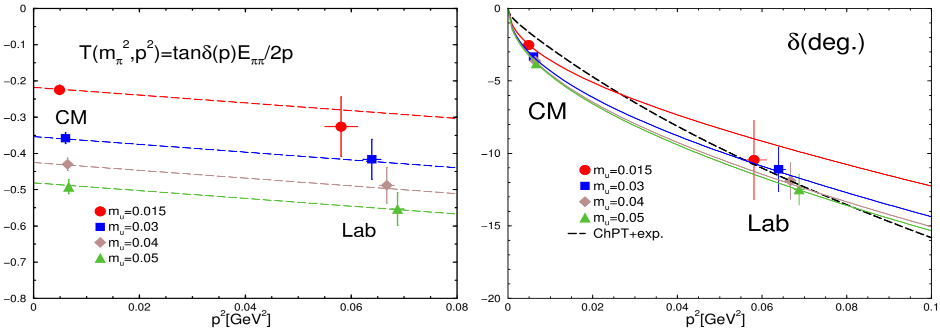

Figure 1: Measured values of T, defined in eq. (3.1), and δ obtained from CM and Lab calculations. Solid lines are fitting results.

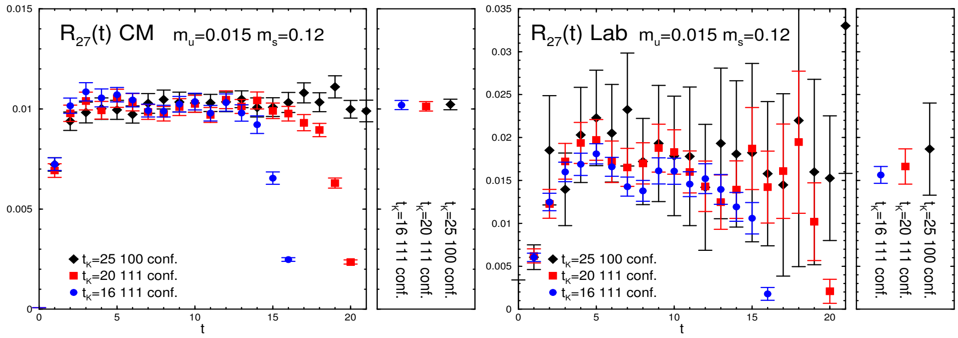

Figure 2: $R_{27}(t)$, defined in eq. (3.2), with different t_K obtained from CM and Lab system calculations. Small figures show averaged values in flat regions.

where $G_{\pi\pi}(t)$ and $G_K(t)$ are $I = 2$ two-pion four-point function and kaon two-point function, respectively, and $Z_{\pi\pi}$ and Z_K are the overlap of the relevant operator with each state. The factor $\sqrt{3}$ comes from changing basis to the isospin basis. When these correlation functions are dominated by each ground state, the ratio $R_i(t)$ will be a constant for those values of t. We then determine the off-shell amplitude in this region.

The figure 2 shows $R_{27}(t)$ for the lightest pion mass and fixed strange quark mass obtained from the CM and Lab system calculations. We plot the ratios with three different kaon source points t_K in the figure. In the CM system, the ratios with different t_K are reasonably flat and consistent with each other in a region $t_\pi \ll t \ll t_K$. The off-shell amplitudes are determined from averaged values in the flat region, whose values are presented in small figure. The three averaged values are also consistent with each other. In the Lab system case, the ratios are noisier than those in the CM case. While the error with $t_K = 25$ is very large, the error decreases as t_k decreases. Thus, we choose the results with $t_K = 16$ in the following analysis, because the result is consistent with the others, and has the smallest error of the three cases. We determine the on-shell amplitude in finite volume by a linear interpolation of the off-shell amplitude with different strange quark masses at fixed light quark mass. The decay amplitudes in infinite volume are obtained for the extended formula of eq. (2.1) by combining the on-shell amplitude and the derivatives of δ and $\phi_{\vec{P}}$.

The matching factors to determine the weak matrix elements were previously calculated in Ref. [4] with the renormalization independent (RI) scheme by a non-perturbative method at the

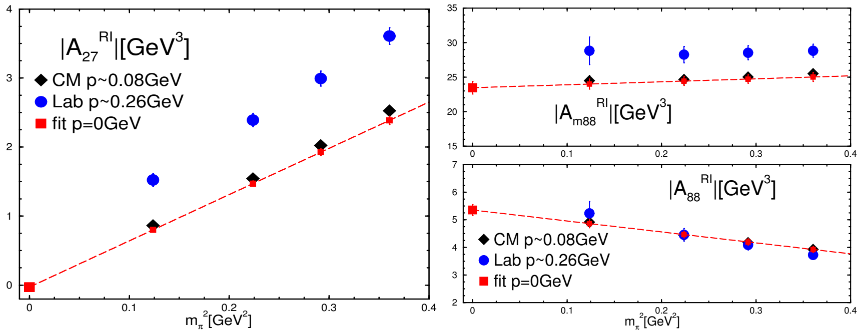

Figure 3: Weak matrix elements obtained from CM and Lab system calculations. Squre symbols are fitting results with $p = 0$.

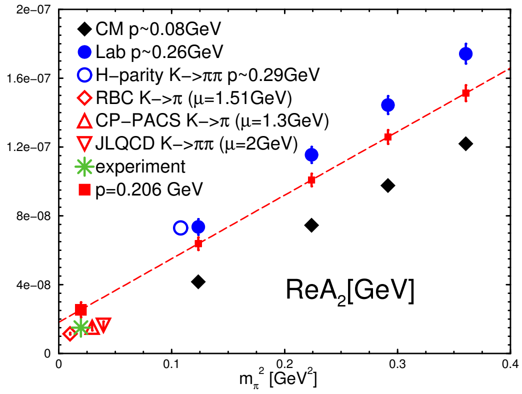

Figure 4: Measured ReA_2 and result from previous works. Square symbols denote fitting results.

scale $\mu = 1.44$ GeV. The weak matrix elements $|A_i^{\rm RI}|$ for $i = 27, 88$ and $m88$ in the RI scheme are plotted in Fig. 3. The weak matrix element of the 27 operator seems to vanish in the chiral limit with $p^2 = 0$, while the other elements remain a constant in the limit. These trends in the pion mass dependence are reasonably consistent with the prediction of ChPT [10]. In order to investigate the m_π and p dependences, we carry out a global fitting for each matrix element for m_π^2 and p^2 assuming a simple polynomial form as

$$A_{00} + A_{10} m_\pi^2 + A_{11} m_\pi^2 p^2 + A_{01} p^2. \tag{3.3}$$

The fitting results at zero momentum for each pion mass and at the chiral limit are represented by square symbols in the figures. The constant in the fitting result A_{00} for the 27 operator is consistent with zero within the error, $A_{00}^{27} = -0.028(25)$, as expected.

We calculate ReA_2 from the weak matrix elements with the Wilson coefficients evaluated by NDR scheme, which is converted to RI scheme at the scale $\mu = 1.44$ GeV. The ReA_2 obtained from the CM and Lab system calculations is shown in Fig. 4. We also plot the results previously obtained with H-parity boundary calculation [4], reduction method [11], and direct calculation with ChPT [12]. The results strongly depend on the pion mass, and also the momentum. We estimate ReA_2 at the physical point, $m_\pi = 0.14$ GeV, $m_K = 0.498$ GeV and $p = 0.206$ GeV, by a global fit making the same polynomial assumption as for the case of the weak matrix elements in eq. (3.3). The square symbols in the figure denote the fitting results at $p = 0.206$ GeV. The ReA_2 at the physical point is $2.54(43) \times 10^{-8}$ GeV which is 1.69(28) times larger than the experiment. In order to understand the difference from the experiment, we need a more detailed investigation of the systematic errors, *e.g.*, finite volume effects, discretization errors, and quenching effect.

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