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Phase transitions and non-analyticities in large N_c gauge theories

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We investigate numerically various phase transitions and non-analyticities at large N using both twisted Eguchi-Kawai space-time reduction and the standard Wilson theory.

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1. Introduction

Despite the considerable simplifications that occur when one takes the 't Hooft limit of pure SU(N) gauge theories, the resulting theory is complex enough to allow a very rich and interesting physics. The phase structure of large N lattice gauge theory on the 4-torus is a good example of this richness. Several transitions were identified and studied in the last few years, *e.g.* the 'bulk' transition [1], the finite temperature deconfining transition [2] and the recently discovered transitions between phases with a different number of broken directions [3]; a recently conjectured non-analyticity associated with the physical size of Wilson loops might also join this list [3].

The purpose of this paper is to present some preliminary results of a numerical study on the phase structure of 4D Euclidian pure SU(∞) gauge theory.

2. TEK model

A very useful alternative to the usual approach of large N extrapolations in lattice gauge theory [4] is the idea of space-time reduction [5]. Due to factorization, the Schwinger-Dyson (loop) equations for infinite- and finite-volume lattice gauge theories coincide in the large N limit. This means that space-time degrees of freedom become spurious at large N, so that the properties of the SU(∞) gauge theory can be analyzed with very small lattices (*in extremis* with an one-point lattice), as long as the gauge group is large enough. This trade between space-time and color degrees of freedom allows us to analyze very large gauge groups without great computational effort.

The *twisted Eguchi-Kawai* (TEK) model is a particular case of a reduced model, consisting of a SU(N) gauge theory defined on a twisted 1^4 lattice. It should reproduce a SU(N) gauge theory defined on a periodic L^4 lattice with the standard Wilson action up to $O(\frac{1}{N^2})$ corrections. The TEK action is given by

$$S_{\text{TEK}}(U) = bN \sum_{\mu>\nu}^{4} \text{Tr}\left(Z_{\mu\nu} U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger + \text{h.c.}\right), \tag{2.1}$$

where $b = \frac{1}{g^2 N}$ is the inverse 't Hooft coupling, $Z_{\mu\nu} = \exp\left(\frac{2\pi i}{N} n_{\mu\nu}\right)$ is the twist, $n_{\mu\nu}$ is an anti-symmetric tensor chosen to be $n_{\mu\nu} = L$ for all $\mu > \nu$, and $L^4 = N^2$ [6]. In addition to the usual gauge symmetry, $U_\mu \mapsto \Omega U_\mu \Omega^\dagger$ ($\Omega \in \text{SU}(N)$), the TEK action (2.1) also has a global Z_N^4 symmetry, $U_\mu \mapsto z_\mu U_\mu$ ($z_\mu \in Z_N$), which is unbroken for all values of the coupling b. The observables of the TEK model are obtained from the ones in the standard Wilson theory by making the substitution $U_\mu(x) \mapsto D(x) U_\mu D(x)^\dagger$, where $D(x) = \prod_{\mu=1}^{4} \Gamma_\mu^{x_\mu}$ and Γ_μ are the vacuum matrices that extremize eqn(2.1). In particular, the reduced Wilson and Polyakov loops are of the form, respectively,

$$W(I,J) = \frac{1}{N} \text{Tr}\, Z_{\mu\nu}^{IJ} U_\mu^I U_\nu^J U_\mu^{\dagger I} U_\nu^{\dagger J}, \tag{2.2}$$

$$P_\mu = \frac{1}{N} \text{Tr}\, U_\mu^L \,. \tag{2.3}$$

The equivalence between the standard Wilson theory and the TEK model states that the expectation values of gauge-invariant observables in these models should coincide up to $O(\frac{1}{N^2})$ corrections:

$$\langle \mathscr{O}[U] \rangle_{\text{W}} = \langle \mathscr{O}[DUD^\dagger] \rangle_{\text{TEK}} + O\left(\frac{1}{N^2}\right). \tag{2.4}$$

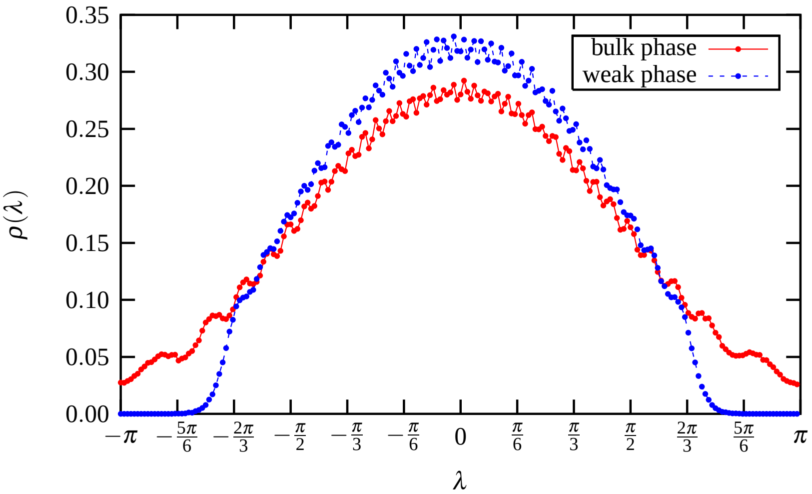

Figure 1: Effect of the 'bulk' transition on the eigenvalue spectrum of the plaquette, simulated in the SU(36) TEK model at fixed coupling $b = 0.3540$: a spectral gap forms in the 'weak-coupling' phase.

3. Wilson loop non-analyticity

The 'bulk' transition is a lattice artifact that affects 4D lattice gauge theories with gauge group $\mathrm{SU}(N \geq 5)$, occurring at a critical coupling $b_c \approx 0.36$ [1]. It is strongly first order and manifests itself by a non-analytic change in the eigenvalue spectrum of the plaquette: in the 'bulk' phase the eigenvalue spectrum is spread over the whole unit circle[1], while in the 'weak-coupling' phase a spectral gap forms around $\lambda = -\pi$ (Fig.1). Recently [3], it was conjectured that a similar transition might occur for larger Wilson loops that would affect their eigenvalue spectrum in a similar way: at $N = \infty$ a spectral gap would form (non-analytically) around $\lambda = -\pi$, for a specific critical value of the coupling that would scale with the *physical size* of the Wilson loop. We looked for this transition in numerical simulations performed in both the standard Wilson theory and the TEK model.

3.1 Method

The method used to detect the formation of a gap in the eigenvalue spectrum of Wilson loops consisted in measuring the changes in the fluctuations of their individual eigenvalues, λ_i. For that purpose we considered the following ratio of correlation functions:

$$R = \frac{\langle \lambda_1^2 \rangle - \langle \lambda_1 \rangle^2}{\langle \lambda_{\frac{N}{2}}^2 \rangle - \langle \lambda_{\frac{N}{2}} \rangle^2} \tag{3.1}$$

This quantity is gauge-invariant, because it only depends on the eigenvalues of the Wilson loop. It corresponds to the fluctuations of the eigenvalue closest to $\lambda = -\pi$, λ_1, normalized by the fluctuations of the one closest to $\lambda = 0$, $\lambda_{\frac{N}{2}}$. For small b the eigenvalue spectrum is spread over the whole interval $[-\pi, \pi]$, without gaps, which leaves enough room for the individual eigenvalues to fluctuate; for large b there is a gap, so the eigenvalues are squeezed and have less room to fluctuate

[1] The eigenvalues of $\mathrm{SU}(N)$ matrices are just phases of the form $e^{i\lambda}$

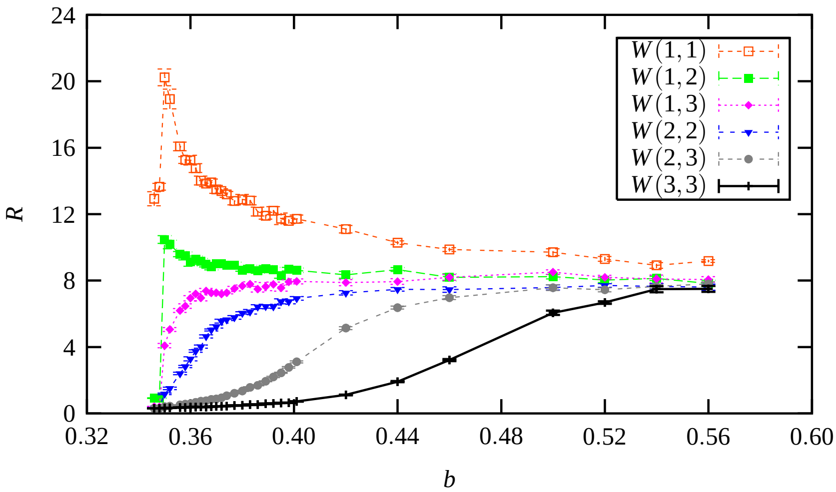

Figure 2: Ratio of correlations R vs coupling b, in the SU(81) TEK model for Wilson loops of several sizes. There is a clear first order 'bulk' transition for the plaquette, $W(1,1)$, as expected. Transitions for larger loops also exist, but they are very smooth.

(Fig.1). The fluctuations of the central eigenvalues like $\lambda_{\frac{N}{2}}$ are in general more affected by this squeezing (*i.e.* decrease faster) then the fluctuations of the outer ones, which results in an increasing of the ratio R across the transition from a gapless to a gapped phase. Therefore, if the formation of a gap in the eigenvalue spectrum of Wilson loops is to be a non-analytical process, we also expect the ratio R to show a sudden jump at the same critical value of the coupling b.

3.2 Results

We performed Monte Carlo simulations to calculate the ratio R of eqn(3.1). We simulated the TEK model with gauge groups $N = 25, 36, 49, 64, 81$ (which correspond to effective lattices of size $L = 5, 6, 7, 8, 9$, respectively); we also simulated the standard Wilson theory with gauge groups $N = 6, 12$ on a 6^4 lattice. The graph in Fig.2 shows the change of R with the coupling b for Wilson loops of several sizes and in the SU(81) TEK model.

The 'bulk' transition, at $b \approx 0.35$, has a very clear non-analytic effect on R for all Wilson loops (especially for the plaquette). For the transitions at larger values of b associated with the formation of a gap in the eigenvalue spectra of large Wilson loops, however, the situation is different: the transitions can be easily seen to exist (Fig.2), but they are very smooth. Justifications for this smooth behavior could be 1) R is not a good candidate for an order parameter of the transition, 2) the transition exists at $N = \infty$, but has large $\frac{1}{N}$ corrections, or 3) there isn't a non-analyticity at all, only a smooth crossover (or it might not even scale correctly with the physical size of the Wilson loops). The answer to these questions is still unknown to us and are being checked. All we can say from these results is that there are in fact transitions, but also that it is very hard to tell how they evolve with N. If one compares how R changes with b for several gauge groups and for a Wilson loop of fixed lattice size, $W(3,3)$ for example, we notice a very slow evolution with increasing N towards a sharper transition (Fig.3). This tendency could result in a non-analyticity at large N, but the data doesn't allow us to reach a definitive conclusion.

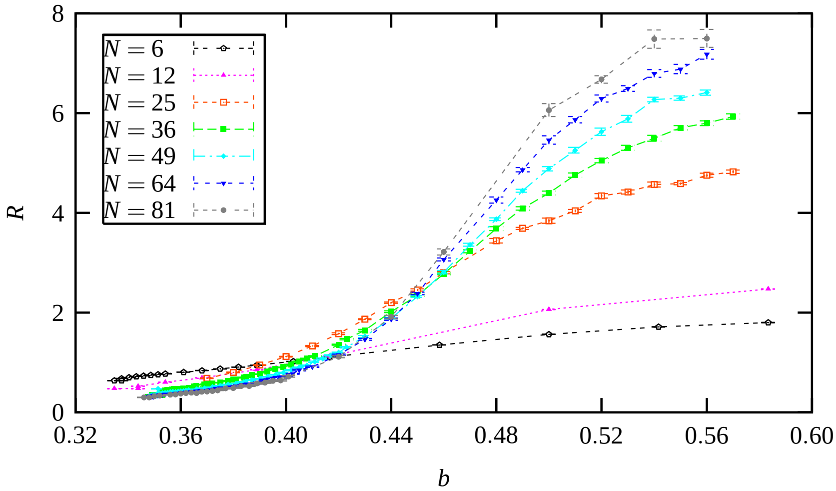

Figure 3: Ratio of correlations R vs coupling b, for Wilson loops of fixed lattice size, $W(3,3)$, and for several gauge groups. The transition shows a slow evolution towards a sharper transition for increasing N, but the data doesn't allow us to reach a definitive conclusion.

4. Polyakov loop transitions

The Polyakov loop is a gauge-invariant observable that winds around a non-contractible loop of the 4-torus. In the standard Wilson picture, its expectation value $\langle P_\mu \rangle_{\mathrm{W}}$ monitors the deconfining transition in pure SU(N) gauge theories, serving as its order parameter. This transition is associated with the spontaneous breaking of a global Z_N symmetry.

The symmetry group, Z_N, acts on Polyakov loops in the following way:

$$P_\mu \mapsto z P_\mu \quad (z \in Z_N). \tag{4.1}$$

This action consists in multiplying all links in a given layer of the lattice, $\{U_\mu(x) | \mu\text{=constant}, x^\mu\text{=constant}\}$, with an element $z \in Z_N$; consequently, all Polyakov loops in the μ-direction come multiplied by z. In the confined phase the Z_N symmetry is realized and $\langle P_\mu \rangle_{\mathrm{W}}$ is forced to be zero, while in the deconfined phase that symmetry is spontaneously broken and there are N possible deconfined phases, $\langle P_\mu \rangle_{\mathrm{W}} \propto \exp(\frac{2\pi i k}{N}) \in Z_N$, $k = 0, \ldots, N-1$.

In the TEK model, however, the gauge fields live on a 1^4 lattice (there is only one layer of links in each direction) and the reduced Polyakov loops are defined as the Lth power of the same link variable, eqn(2.3). Therefore, the symmetry group that acts on the reduced Polyakov loops is Z_L and not Z_N, due to the Z_N^4 symmetry of the TEK action:

$$U_\mu \mapsto z_\mu U_\mu \;\; \Rightarrow \;\; P_\mu \mapsto z_\mu^L P_\mu \quad (z_\mu \in Z_N \;\; \Rightarrow \;\; z_\mu^L \in Z_L). \tag{4.2}$$

This Z_L symmetry of the reduced Polyakov loops breaks at a given critical coupling, which in all the numerical simulations performed coincided with the ‘bulk’ transition. The corresponding ‘confined’ phase is characterized by $\langle P_\mu \rangle_{\mathrm{TEK}} = 0$, just like in the standard Wilson theory, while in the ‘deconfined’ phase (where the Z_L symmetry is broken) there are *only* $L = \sqrt{N}$ possible deconfined phases, $\langle P_\mu \rangle_{\mathrm{TEK}} \propto \exp(\frac{2\pi i k}{L}) \in Z_L$, $k = 0, \ldots, L-1$. This property reveals an explicit difference between the TEK reduced models and their equivalent standard Wilson theory: they haven't the same number of Polyakov loop phases, contrary to what one might naïvely expect from eqn(2.4). This observation might lead to the question of whether the usual argument of equivalence between TEK models and standard Wilson theory using lattice Schwinger-Dyson equations at large N also applies to non-contractible loops, and consequently if a study of the deconfining transition using reduced Polyakov loops is valid.

5. Conclusions

We presented some preliminary results of a numerical study on the phase structure of large N gauge theories on the 4-torus. We showed evidence of the existence of transitions associated with a gap formation in the eigenvalue spectrum of large Wilson loops. These transitions, however, are very smooth and their evolution with increasing N does not allow conclusions about their nature at $N = \infty$. To conclusively infer from numerical calculations the existence (or not) of the Wilson loop non-analyticity discussed in section 3 we might need to simulate larger gauge groups ($N > 81$), for which the TEK model is the only relevant and practicable method.

We also checked the behavior of reduced Polyakov loops in the TEK model. It can be easily seen from symmetry arguments that reduced Polyakov loops behave differently from their standard Wilson theory equivalents, contrary to what one might naïvely expect from eqn(2.4). Therefore, the transitions of reduced Polyakov loops might not be a good tool for studying the large N deconfining transition.

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References

[1] M. Campostrini, *Nucl. Phys. Proc. Suppl.* **73** (1999) 724.

[2] B. Lucini, M. Teper and U. Wenger, *Phys. Lett.* **B545** (2002) 197 [hep-lat/0206029]; *JHEP* **0406** (2004) 061 [hep-lat/0307017].

[3] J. Kiskis, R. Narayanan and H. Neuberger, *Phys. Lett.* **B574** (2003) 65 [hep-lat/0308033]; R. Narayanan and H. Neuberger, *Phys. Rev. Lett.* **91** (2003) 081601 [hep-lat/0303023]; H. Neuberger, *Phases of planar QCD on the torus*, in proceedings of *XXIIIrd International Symposium on Lattice Field Theory*, PoS(LAT2005)005 [hep-lat/0509014].

[4] M. Teper, *Large N*, in proceedings of *XXIIIrd International Symposium on Lattice Field Theory*, PoS(LAT2005)256 [hep-lat/0509019].

[5] T. Eguchi and H. Kawai, *Phys. Rev. Lett.* **48** (1982) 1063; S. R. Das, *Rev. Mod. Phys.* **59** (1987) 235.

[6] A. González-Arroyo and M. Okawa, *Phys. Rev. D* **27** (1983) 2397.