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The study of SU(3) super Yang-Mills quantum mechanics.

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We present the hamiltonian study of super Yang-Mills quantum mechanics (SYMQM). The recently introduced method based on Fock space representation allows to analyze SYMQM numerically. The detailed analysis for SYMQM in two dimensions for $SU(3)$ group is given.

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1. Introduction

Supersymmetric Yang-Mills quantum mechanics by definition are $\mathcal{N}=1$ super Yang-Mills field quantum theories reduced from $D=d+1$ to $D=0+1$ dimensions [1]. Supersymmetry requires the space-time dimension to be $D=2,4,6,10$ with $\mathcal{N}=2,4,8,16$ supercharges in the resulting quantum mechanics respectively. The gauge connection A_μ after dimensional reduction becomes bosonic coordinate x_a^i, $i=1,\dots,d$, $a=1,\dots,N^2-1$, in the adjoint representation of $SU(N)$. We will denote their conjugate momenta by p_a^i, $[x_a^i, p_b^j]=i\delta_{ab}\delta^{ij}$. The hamiltonian is then

$$H=\frac{1}{2}p_a^i p_a^i+\frac{1}{4}g^2(f_{abc}x_b^i x_c^j)^2+H_F, \tag{1.1}$$

where $H_F=-\frac{i}{2}g f_{abc}\vartheta_a^\alpha x_b^i \Gamma^i_{\alpha\beta}\vartheta_c^\beta$ for $D=3,10$ and ϑ are real spinors obeying $\{\vartheta_a^\alpha,\vartheta_b^\beta\}=\delta^{\alpha\beta}\delta_{ab}$, $\alpha,\beta=1,\dots,\mathcal{N}$ or $H_F=ig f_{abc}\bar{\vartheta}_a^\alpha x_b^i\Gamma^i_{\alpha\beta}\vartheta_c^\beta$ for $D=4,6$ and ϑ are complex spinors obeying $\{\bar{\vartheta}_a^\alpha,\vartheta_b^\beta\}=\delta^{\alpha\beta}\delta_{ab}$ for $\alpha,\beta=1,\dots,\frac{\mathcal{N}}{2}$. The $\Gamma^i_{\alpha\beta}$ are matrix representation of an $SO(d)$ Clifford algebra $\{\Gamma^i,\Gamma^j\}=2\delta^{ij}$. The rational symmetry of the original theory become now the internal Spin(d) symmetry. The constraints for A_0 component of gauge connection require all the physical states to be $SU(N)$ singlets. It is also the condition for the supersymmetry i.e. hamiltonian (1.1) is supersymmetric only in gauge invariant sector.

The bosonic part of H was firstly discovered in pure Yang-Mills theory on lattice in the zero volume limit [2]. Later on it appeared as a system describing quantum mechanics of a membrane moving in d dimensional space and its supersymmetric extension i.e. the supermembrane [3]. The reason why these models have been so extensively studied recently is the BFSS (Banks, Fischler, Shenker, Susskind) conjecture [4] where it was argued that in $N\to\infty$ limit Eg.(1.1) describes M-theory in infinite momentum frame although the covariant formulation of the latter is still lacking. The detailed study of the hamiltonian H shows that in bosonic sector the potential is confining and there is no continuous spectrum [6]. If however the supersymmetry is turned on then in fermion rich sectors there are bound states as well as scattering ones [7]. The only solutions existing in the literature are for D=1+1, $SU(2)$ [1] and its generalization for arbitrary $SU(N)$ [5]. In this paper we focus on these two dimensional systems using the numerical approach described in section 2. In section 3 we give some general properties of $SU(N)$ models and present the detailed results for $SU(3)$ group.

2. Cutoff method

The cutoff method [8] is conceptually very simple. First of all we introduce bosonic and fermionic creation and annihilation operators $a^\dagger{}_a$, a_a, $f_a^\dagger$, f_a [1] i.e.

$$a_a=\frac{1}{\sqrt{2}}(x_a+ip_a),\quad f_a=\vartheta_b,\quad [a_a,a_b^\dagger]=\delta_{ab},\quad \{f_a,f_b^\dagger\}=\delta_{ab}.$$

Next we truncate the Hilbert space to the maximal number of quanta

$$n_B=\sum_b a_b^\dagger a_b,\quad n_B\le n_{Bmax},$$

[1] Since we discuss only two dimensional models the spatial indices are omitted.

compute matrix elements of H and diagonalize the resulting finite matrix. In this way one can analyze the n_{Bmax} dependance of spectrum. There is a big difference between continuous and discrete spectrum behavior with cutoff namely

$$E_m^{n_{Bmax}} = E_m + O(e^{-n_{Bmax}}) \ - \ \text{discrete spectrum}$$

$$E_m^{n_{Bmax}} = O(\frac{1}{n_{Bmax}}) \ - \ \text{continuous spectrum}$$

where m is an index of the energy level $m = 1,\dots,n_{Bmax}+1$. The limit $n_{Bmax} \longrightarrow \infty$ is called the continuum limit. In the discrete spectrum case the energy levels converge rapidly to the exact eigenvalues of the hamiltonian. This may not be surprising however it is interesting to see how fast the convergence is. For details the reader is referred to [9]. If the spectrum is continuous the behavior is different. The convergence is very slow and all the eigenvalues vanish in infinite cutoff limit. In the continuum limit the spectrum is supposed to be continuous therefore we have to introduce the following scaling law [10]

$$m(n_{Bmax}) = const.\sqrt{n_{Bmax}} \Longleftrightarrow E^{n_{Bmax}}_{m(n_{Bmax})} \rightarrow E\,.$$

It was claimed in [10] that this scaling law should work independently of the theory whenever one can define scattering states asymptotically. The argument for the above claim is based on the following fact. The eigenvalues of the momentum operator in ordinary d=1 quantum mechanics in cut Fock space are zeros of Hermite polynomials $H_{n_B}(x)$ the asymptotic behavior of which is $\frac{1}{\sqrt{n_B}}$ [9,10]. Therefore once the momentum operator is defined its spectrum cutoff dependance should be $\frac{1}{\sqrt{n_B}}$ for large n_B.

3. Two dimensional SYMQM

The hamiltonian in $D=2$ reduces to $H=\frac{1}{2}p_a p_a$ in a gauge invariant sector. It is free however non trivial due to the gauge constraint. Since we are now working in $SU(N)$ it is evident that any gauge invariant state has to be of the form

$$T_{bc\dots de\dots} a_b^\dagger a_c^\dagger \dots f_d^\dagger f_e^\dagger \dots |\,0\rangle, \tag{3.1}$$

where $T_{bc\dots de\dots}$ is some $SU(N)$ invariant tensor made out of structure tensors $f_{abc}, d_{abc}, \delta_{ab}$. There are plenty of identities between the later hance many states of the form (3.1) will be linearly dependant. We therefore ask how does the basis in gauge invariant sector look like when working with structure tensors? In this section we present a very helpful diagrammatic method which gives us the way to answer this question.

3.1 Birdtracs

In order to deal with the variety of all possible tensor contractions we introduce the diagrammatic approach (figure 1).

Each leg corresponds to one index and summing over any two indices is simply gluing appropriate legs. Structure tensors f_{ijk}, d_{ijk} are represented by vertices and δ_{ij} is a line. Any tensor is

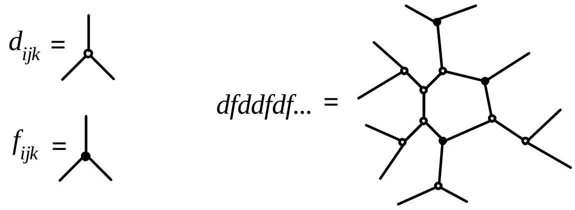

Figure 1: Diagrammatic notation of invariant tensors.

now represented by a graph. Such diagrammatic approach has already been introduced long time ago by Cvitanovič [11]. In general one can construct loop tensor which by definition is a tensor that in diagrammatic notation looks like a loop. It can be proved [12] that any such loop can be expressed in terms of forests i.e. products of tree tensors (figure 2).

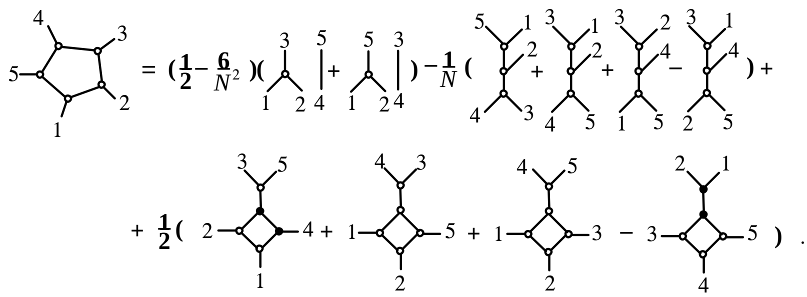

Figure 2: An example of loop reduction for pentagon made out of d_{ijk} tensors.

Therefore we are left with tree tensors only. These however can be easily expressed in terms of trace tensors $Tr(T_aT_b\ldots)$ where T_a are $SU(N)$ generators in fundamental representation. With the use of the following matrices $A^\dagger = a_b^\dagger T_b$, $F^\dagger = f_b^\dagger T_b$ any gauge invariant state is an appropriate linear combination of products of trace states

$$Tr(A^{\dagger^{i_1}} F^\dagger A^{\dagger^{i_2}} F^\dagger \ldots A^{\dagger^{i_k}} F^\dagger) \mid 0\rangle.$$

Due to the grassmann algebra the number of F matrices under the trace cannot be grater then N^2-1 i.e. $k \leq N^2-1$. Moreover the Cayley-Hamilton theorem for A matrices gives $i_k \leq N$. The remaining set of states is still linearly dependent and the further analysis requires separate study of each $SU(N)$. The basis states in F=0 sector are of the form

$$\mid i_2, i_3, \ldots, i_N\rangle = Tr^{i_2}(A^\dagger A^\dagger) Tr^{i_3}(A^\dagger A^\dagger A^\dagger) \ldots Tr^{i_N}(A^\dagger \ldots A^\dagger) \mid 0\rangle.$$

We see that there are as many states with given number of quanta n_B as there are natural solutions of the equation $2i_2 + 3i_3 + \ldots + Ni_N = n_B$. For $U(N)$ this would be exactly $p(n_B)$ - the partition number of n_B. For $SU(N)$ this is a little less the $p(n_B)$ however it still grows exponentially with $\sqrt{n_B}$.

In order to compute the spectrum of the hamiltonian in bosonic sector one has to compute the following scalar product

$$S^{i_2\ldots i_N}_{j_2\ldots j_N} = \langle i_2 \ldots i_N \mid j_2 \ldots j_N\rangle,$$

which in principle is a tedious task but not impossible.

We now discuss the "bilinear limit" i.e. the limit of the following restricted $SU(N)$ basis

$$\mid 2n\rangle = Tr^n(A^\dagger A^\dagger) \mid 0\rangle, \tag{3.2}$$

which was introduced by Lüscher in [2] only in D=3+1 case. In this basis the non zero hamiltonian matrix elements are easy to derive which will be discussed elsewhere. Therefore it is straightforward to proceed with the cutoff analysis (figure 3).

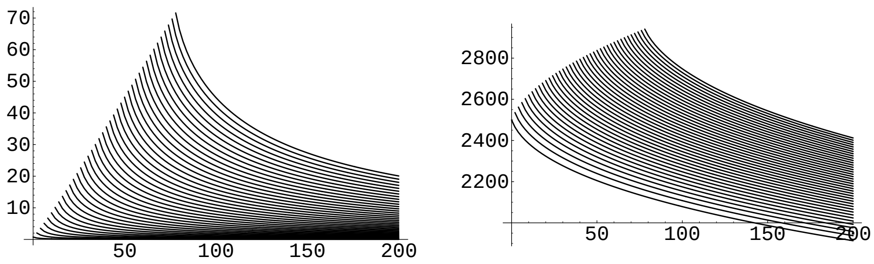

Figure 3: The cutoff dependance of spectrum for $SU(2)$ and $SU(100)$ in Lüscher basis.

We see that there is no quantitative difference between $SU(2)$ and eg. $SU(100)$ case. This is not what we have expected and it means that the restricted basis (3.2) simplifies too much.

3.2 $D = 1+1$, $SU(3)$ SYMQM

Here we briefly present the calculations of Hamiltonian matrix elements in bosonic sector. The basis and the scalar products we are interested in are know

$$\mid i, j\rangle = Tr^i(A^\dagger A^\dagger) Tr^j(A^\dagger A^\dagger A^\dagger) \mid 0\rangle, \qquad S^{i\ j}_{i'\ j'} = \langle i, j \mid i', j'\rangle.$$

The only non vanishing elements of $S^{i\ j}_{i'\ j'}$ are the ones obeying the constraint $2i+3j = 2i'+3j'$. Therefore it is convenient to work with the following symbol

$$W^k_{i\ j} = \langle i, j \mid Tr^{2k}(AAA) Tr^{3k}(A^\dagger A^\dagger) \mid i, j\rangle,$$

which has the advantage of reproducing only the non vanishing $S^{i\ j}_{i'\ j'}$'s. It is tedious but possible to obtain formulas and recurrence equations for W^k_{ij} hance one can, at least on the computer, derive the exact $S^{i\ j}_{i'\ j'}$ values. This S matrix is in fact the Gram matrix therefore its non block diagonal form signals that we still have to orthogonalize the basis. We will not do so however. In order to represent the hamiltonian H in orthogonal basis we follow [13]. It is sufficient to calculate the gram matrix G and proceed with the following similarity transformation

$$H_{ort} = G^{-\frac{1}{2}} H G^{-\frac{1}{2}}.$$

The results of the cutoff analysis are presented in figure 4. It is clear that the spectrum seems to be far more complicated than in $SU(2)$ case.

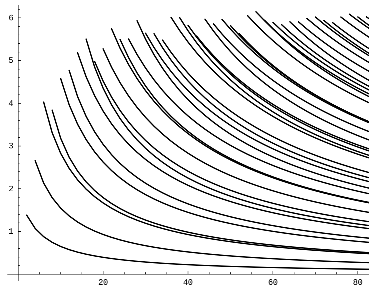

Figure 4: The cutoff dependance of spectrum in $D = 1+1$, $SU(3)$, $F = 0$.

4. Summary

SYMQM models seem to reveal verity of application in several areas of physics (Yang-Mills theories, supersymmetry, strings) hance their detailed analysis is of interest. Although they are rich in symmetries ($SU(N)$, $SO(d)$, supersymmetry) the exact solutions are missing in the literature forcing one to apply numerical methods. The cutoff method presented here is working surprisingly well. The results in two dimensional systems are very encouraging and give a hope to proceed with the $N \to \infty$ limit especially when exact solutions in this case are known.

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