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Lattice simulations of Born-Infeld non-linear QED

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Born-Infeld non-linear electrodynamics was introduced to render the self energy of a point particle finite. It has recently been revived as a field theory for branes and strings. We quantize this theory on a Euclidean space-time lattice, using Metropolis Monte-Carlo simulations to measure the properties of the quantum field theory. Lüscher-Weisz methods are used to measure the electromagnetic fields from a static point charge. The **D** field from a point charge appears to be identical to that for the normal Maxwell Lagrangian. The **E** field is enhanced by quantum fluctuations, and shows short distance screening as it does in the classical theory.

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1. Introduction

The $n+1$ dimensional Born-Infeld (non-linear electrodynamics) action [1, 2] is:

$$S = b^2 \int d^{n+1}x \left[1 - \sqrt{-\det\left(g_{\mu\nu} + \frac{1}{b}F_{\mu\nu}\right)}\right]. \tag{1.1}$$

This has seen a revival as a theory of strings and branes [3, 4, 5, 6, 7]. Choosing $n = 9$, and dimensionally reducing this action from $9+1$ to $p+1$ dimensions describes a p-brane. The $9-p$ additional components of A_μ are identified with the transverse components of the string/brane.

Most of the serious work on these theories has dealt with their classical behaviour [8]. We are simulating these quantum theories on the lattice. We are starting with the simplest case where $n = p = 3$, the original Born-Infeld modification of electrodynamics, designed to make the self energy of a point charge finite.

Section 2 reviews the classical Born-Infeld theory. In section 3 we indicate how this is ported to the lattice allowing Monte-Carlo simulations of the quantum theory. Section 4 details our simulations and preliminary results. A discussion of our results and conclusions are given in section 5.

2. Classical Born-Infeld electrodynamics in Minkowski space-time

This section summarises those results in references [9, 10, 11, 12] which are relevant for our investigations.

Evaluating the determinant in equation 1.1 the Lagrangian in $3+1$ dimensions is

$$\mathscr{L} = b^2[1 - \sqrt{1 - b^{-2}(\boldsymbol{E}^2 - \boldsymbol{B}^2) - b^{-4}(\boldsymbol{E}\cdot\boldsymbol{B})^2}]. \tag{2.1}$$

One can now define $\boldsymbol{D}$ and $\boldsymbol{H}$ by

$$\begin{aligned}\boldsymbol{D} &= \frac{\partial \mathscr{L}}{\partial \boldsymbol{E}} = \frac{\boldsymbol{E} + b^{-2}(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{B}}{\sqrt{1 - b^{-2}(\boldsymbol{E}^2 - \boldsymbol{B}^2) - b^{-4}(\boldsymbol{E}\cdot\boldsymbol{B})^2}}\\ \boldsymbol{H} &= \frac{\partial \mathscr{L}}{\partial \boldsymbol{B}} = \frac{\boldsymbol{B} - b^{-2}(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{E}}{\sqrt{1 - b^{-2}(\boldsymbol{E}^2 - \boldsymbol{B}^2) - b^{-4}(\boldsymbol{E}\cdot\boldsymbol{B})^2}}.\end{aligned} \tag{2.2}$$

Interaction with charged particles is implemented, as usual, by adding a term $j_\mu A^\mu$ to the Lagrangian. In terms of $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{D}$ and $\boldsymbol{H}$, the equations of motion are the standard Maxwell equations. The non-linearity is hidden in equations 2.2.

For a static point charge $\rho = e\delta^3(\boldsymbol{r})$ the electric fields are

$$\begin{aligned}\boldsymbol{D} &= \frac{e}{4\pi r^2}\hat{\boldsymbol{r}}\\ \boldsymbol{E} &= \frac{e}{4\pi}\frac{\hat{\boldsymbol{r}}}{\sqrt{r^4 + r_0^4}},\end{aligned} \tag{2.3}$$

where $\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{r}$ and $r_0 = \sqrt{\frac{|e|}{4\pi b}}$. Hence the $\boldsymbol{D}$ field for a static point charge is identical to the Maxwell solution, while the $\boldsymbol{E}$ field is screened at short distances.

3. Lattice Born-Infeld quantum-electrodynamics

The Euclidean space action for Born-Infeld QED

$$S = b^2 \int d^4x[\sqrt{1 + b^{-2}(\boldsymbol{E}^2 + \boldsymbol{B}^2) + b^{-4}(\boldsymbol{E} \cdot \boldsymbol{B})^2} - 1] \tag{3.1}$$

is positive. Hence it can be simulated using Monte-Carlo methods.

On the lattice we use the non-compact formulation:

$$F_{\mu\nu}(x + \tfrac{1}{2}\hat{\mu} + \tfrac{1}{2}\hat{\nu}) = A_\nu(x + \hat{\mu}) - A_\nu(x) - A_\mu(x + \hat{\nu}) + A_\mu(x) \tag{3.2}$$

and average over the 16 choices of 6 plaquettes associated with each lattice site. We also define $\beta = b^2 a^4$ where a is the lattice spacing. Simulations are performed using the Metropolis Monte-Carlo method [13].

We measure the $\boldsymbol{E}$ and $\boldsymbol{D}$ fields due to a static point charge. This point charge e is introduced by including a Wilson Line (Polyakov Loop) $W(\boldsymbol{x})$.

$$W(\boldsymbol{x}) = \exp\left\{ ie\sum_t \left[A_4(\boldsymbol{x},t) - \frac{1}{N_x N_y N_z}\sum_{\boldsymbol{y}} A_4(\boldsymbol{y},t) \right] \right\} \tag{3.3}$$

The second ('Jellium') term is needed, since a net charge would be inconsistent with periodic boundary conditions on A_μ. $\langle \boldsymbol{E} \rangle$ and $\langle \boldsymbol{D} \rangle$ in the presence of this charge are given by

$$\begin{aligned} i\langle \boldsymbol{E} \rangle_\rho(\boldsymbol{y} - \boldsymbol{x}) &= \frac{\langle \boldsymbol{E}(\boldsymbol{y},t) W(\boldsymbol{x}) \rangle}{\langle W(\boldsymbol{x}) \rangle} \\ i\langle \boldsymbol{D} \rangle_\rho(\boldsymbol{y} - \boldsymbol{x}) &= \frac{\langle \boldsymbol{D}(\boldsymbol{y},t) W(\boldsymbol{x}) \rangle}{\langle W(\boldsymbol{x}) \rangle}. \end{aligned} \tag{3.4}$$

Since W is complex, there is a sign problem, which causes $\langle W(\boldsymbol{x}) \rangle$ to fall exponentially with N_t. We use the method of Lüscher and Weisz [14] (Parisi, Petronzio and Rapuano [15]) with thickness 1 and 2 timeslices to overcome this exponential factor.

4. Simulations and Results

We have performed preliminary simulations of 500,000 10-hit Metropolis sweeps of the lattice at $\beta = 100, 1.0, 0.01, 0.0001$ and 100,000 sweeps at $\beta = 5, 2, 0.5, 0.2, 0.1$, making measurements every 100 sweeps. We measured the $\boldsymbol{E}$ and $\boldsymbol{D}$ fields for on axis separations from the point charge. Figure 1 shows the expectation value of the Wilson lines (Polyakov loops) obtained from these simulations. Note that the value falls rapidly with increasing e. The rate of falloff also increases with increasing non-linearity (decreasing β). Note also the small relative errors, even when the magnitude has fallen 8 orders of magnitude, which shows the effectiveness of the Lüscher-Weisz method.

Figure 2 shows the ratio of the $\boldsymbol{E}$ field in the direction of the separation from the charge to the $\boldsymbol{D} = \boldsymbol{E}$ field for the free field (Maxwell) theory, at the minimum separation ($Z = 0.5$), in the limit of zero charge. At large β, where the Born-Infeld theory asymptotes to the Maxwell theory, this ratio

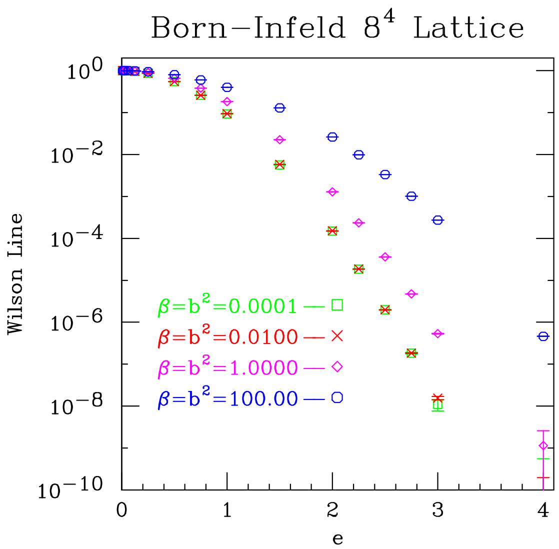

Figure 1: Wilson Lines as functions of charge e, for a range of $\beta = a^2 b^2$.

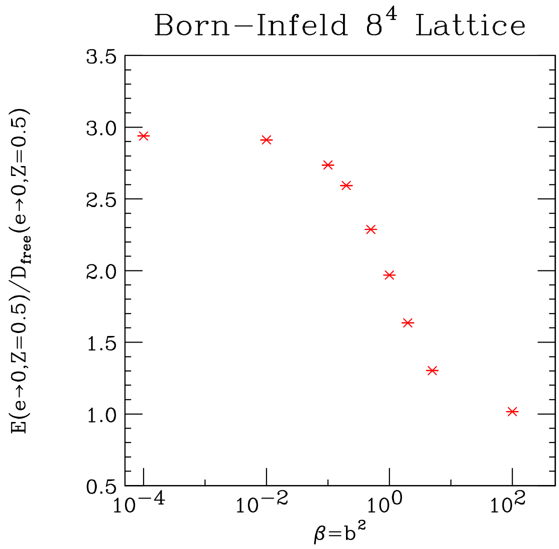

Figure 2: E/D ratio at minimum separation for $e \to 0$ as a function of β.

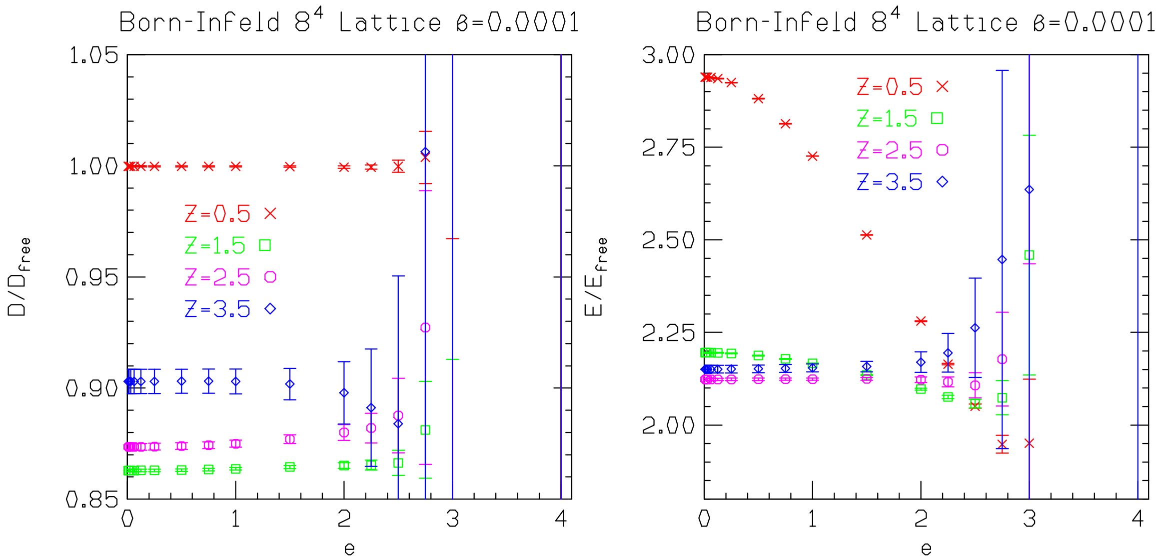

Figure 3: a) $\boldsymbol{D}$ fields at distance Z from the point charge e, scaled by the free field (Maxwell) values, for $\beta = 0.0001$. b) $\boldsymbol{E}$ fields at distance Z from the point charge e, scaled by the free field (Maxwell) values, for $\beta = 0.0001$.

approaches 1. Classically, this ratio is 1 for all values of b. Quantum fluctuations cause this ratio to increase with increasing non-linearity (decreasing β), approaching a value close to 3 for small β.

In figure 3a, we plot the $\boldsymbol{D}$ fields scaled by their free field values at each separation, as a function of charge, for $\beta = 0.0001$, where the non-linearity is large, and the $\boldsymbol{D}$ field comes almost entirely from the $(\boldsymbol{E}\cdot\boldsymbol{B})\boldsymbol{B}$ term in its definition. The fact that this ratio is still 1 for all e at minimal separation is because $\boldsymbol{D}$ still obeys $\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho$, combined with cubic symmetry. Because we do not have full rotational symmetry on the lattice $\boldsymbol{\nabla}\cdot\boldsymbol{D} = \rho$ is insufficient to make this ratio 1 at other separations. The fact that this ratio is never more than 15% from 1 suggests that it would be 1 if we had rotational symmetry. However, we would expect rotational symmetry to be restored at large distances, which is why the ratio is closer to 1 for larger separations. Figure 3b is a similar graph for the $\boldsymbol{E}$ field. As well as showing the effects of quantum fluctuations as in figure 2, the $\boldsymbol{E}$ field is clearly screened at short distances as e increases, similar to what is seen in the classical theory.

What is different from the classical theory is that classically the screening length continues to increase with decreasing b. The quantum theory approaches a limit as $\beta \to 0$.

5. Discussion and conclusions

We have succeeded in using lattice Monte-Carlo methods to extract non-perturbative physics from Born-Infeld electrodynamics, quantized using the Euclidean-time functional integral approach. The on-axis (quantum) electrostatic fields of a point charge are measured as functions of the charge e introduced as a Wilson Line. The approach of Lüscher and Weisz, which reduces these measurements from an exponential- to a polynomial-time problem, was essential for extracting these quantities.

In the classical field-theory $E/D \to 1$ as $e \to 0$. For the quantum theory E/D increases from 1 as the nonlinearity is increased indicating that the dielectric constant $\varepsilon < 1$.

As $|e|$ is increased, the $\boldsymbol{E}$ field is screened at short distances. Screening increases with $|e|$ and with increasing nonlinearity (β or b). The screening length r_0 appears to increase as $\sqrt{|e|}$ as for the classical theory. $\boldsymbol{D}$ shows no such screening and appears to independent of the non-linearity, while $\boldsymbol{D}/e$ appears independent of e.

Unlike the classical theory, where the screening length diverges as $b \to 0$, the quantum theory approaches a fixed-point field theory as $\beta = b^2a^4 \to 0$. This conformal field theory has Euclidean Lagrangian $\mathscr{L}_E = \frac{1}{4}|\boldsymbol{E}.\boldsymbol{B}|$ and Hamiltonian $\mathscr{H} = |\boldsymbol{D} \times \boldsymbol{B}|$ [9, 16].

Normally, Born-Infeld QED is considered as an effective field theory with a momentum cutoff. However, as this cutoff $\to \infty$ it approaches the above fixed-point theory. If this fixed-point field theory is non-trivial, it would serve to define Born-Infeld QED without a cutoff.

These first simulations were performed on 8^4 lattices. We are extending these simulations to larger lattices. We then plan to study those p-brane theories obtained by dimensional reduction of $n+1$ dimensional Born-Infeld theories to determine if the quantized theories continue to show string/brane dynamics.

Acknowledgements

Our simulations are currently running on the Rachael supercomputer at the Pittsburgh Supercomputer Center.

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PoS(LAT2005)261

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