QED$_3$ on a space-time lattice: a comparison between compact and noncompact formulation

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Quantum electrodynamics in a (2+1)-dimensional space-time has been object of studies both as effective theory for the pseudogap phase of high-T$_c$ superconductors and for the theoretical investigation of mechanisms of confinement in presence of matter fields. We discretize the theory using both compact and noncompact formulations, analyze the behavior of the chiral condensate and of the monopole density and compare them. Finally we draw some conclusions about the possible equivalence of the two lattice formulations.
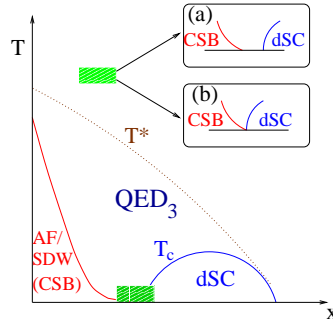

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Figure 1: Phase diagram in the (x, T) plane [12]; here x represents the doping and T the temperature.

1. Introduction

Quantum electrodynamics in 2+1 dimensions (QED$_3$) is an interesting testfield for understanding the mechanism of confinement in gauge theories [2], and for the effective description of low-dimensional, correlated, electronic condensed matter systems, like spin systems [3, 4], or high-T_c superconductors [5]. The compact formulation of QED$_3$ is more suitable for studying the mechanism of confinement, while both compact [6] and noncompact formulations arise in condensed matter systems. This work is devoted to clarify the relationship between these two formulations of QED$_3$ on the lattice (for more details, see Ref. [1]).

Compact QED$_3$ without fermion degrees of freedom is always confining [2]; a charge and anti-charge pair is confined by a linear potential, as an effect of the proliferation of magnetic monopoles. If matter fields are introduced, the interaction between monopoles could turn from $1/x$ to $-\ln(x)$ at large distances x [7], so that the deconfined phase may become stable. However, it has been proposed that compact QED$_3$ with massless fermions is always in the confined phase [8, 9].

At finite T noncompact QED$_3$ is relevant in the analysis of the pseudo gap phase [10] of cuprates. In Fig. 1 we report the phase diagram of high-T_c cuprates. The small-x phase (x is the doping) is characterized [11] by an insulating antiferromagnet (AF); by increasing x, this phase evolves into a spin density wave (SDW), that is a weak antiferromagnet. The pseudo gap phase is located between this phase and the d-wave superconducting (dSC) one. The effective theory of the pseudo gap phase [10] turns out to be QED$_3$ [5, 12, 13], with spatial anisotropies in the covariant derivatives [11], and with Fermionic matter given by spin-$1/2$ chargeless excitations of the superconducting state (spinons). These excitations are minimally coupled to a massless gauge field, which arises from the fluctuating topological defects in the superconducting phase. The SDW order parameter is the chiral condensate $\langle \bar{\psi}\psi \rangle$ [13]. If $\langle \bar{\psi}\psi \rangle$ is different from zero, then the d-wave superconducting phase is connected to the spin density wave one (see Fig. 1 case b); otherwise the two phases are separated at $T = 0$ by the pseudo gap phase (see Fig. 1 case a).

The determination of the critical number of flavors, $N_{f,c}$, such that for $N_f < N_{f,c}$ the chiral condensate is nonzero, is a crucial point addressed in many works (see Ref. [1] for a detailed list). Here, we shall not try to ascertain $N_{f,c}$. However, we will handle the problem of confinement in presence of massless fermions, by looking at the relation between monopole density and fermion mass. Moreover we shall focus on a comparative study of the two lattice formulations of QED$_3$, the

compact and the noncompact ones. In particular, we revisit the analysis of Fiebig and Woloshyn of Refs. [16, 17], where the dynamic equivalence between the two formulations of (isotropic) QED_3 is claimed to be valid in the finite lattice regime.

2. Compact and noncompact formulations

We adopt here the definition of the QED_3 parity-conserving continuum Lagrangian density in Minkowski metric given in Ref. [18],

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}^2 + \overline{\psi}_i iD_\mu \gamma^\mu \psi_i - m_0 \overline{\psi}_i \psi_i , \qquad (2.1)$$

where ψ_i $(i = 1, \ldots, N_f)$ are 4-component spinors. Since QED_3 is a super-renormalizable theory, $\dim[e] = +1/2$, the coupling does not display any energy dependence. For the definition of the γ matrices, for the chiral and parity properties of the theory we refer to Ref. [1].

The lattice Euclidean action [19, 14] using staggered fermion fields $\overline{\chi}, \chi$, is given by

$$S = S_G + \sum_{i=1}^{N} \sum_{n,m} \overline{\chi}_i(n) M_{n,m} \chi_i(m) , \qquad (2.2)$$

where S_G is the gauge field action and $M_{n,m}$ is the fermion matrix. The action (2.2) allows to simulate $N = 1, 2$ flavours of staggered fermions corresponding to $N_f = 2, 4$ flavours of 4-component fermions ψ [20]. For the compact formulation S_G is the standard Wilson action written in terms of the plaquette variable $U_{\mu\nu}(n)$. Instead, in the noncompact formulation one has

$$S_G[\alpha] = \frac{\beta}{2} \sum_{n,\mu<\nu} F_{\mu\nu}(n)F_{\mu\nu}(n) , \quad F_{\mu\nu}(n) = \{\alpha_\nu(n+\hat{\mu}) - \alpha_\nu(n)\} - \{\alpha_\mu(n+\hat{\nu}) - \alpha_\mu(n)\} , \quad (2.3)$$

where $\alpha_\mu(n)$ is the phase of the link variable $U_\mu(n)$ and $\beta = 1/(e^2 a)$.

Monopoles are detected in the lattice using the method given by DeGrand and Toussaint [21]. In the noncompact formulation of QED_3 monopoles are not classical solutions as in compact QED_3, but they could give a contribution to the Feynman path integral owing to the periodic structure of the fermionic sector [22].

As a signal for continuum physics, we look for plateau of dimensionless observables, such as $\beta^2 \langle \overline{\chi}\chi \rangle$. There are two regimes: for β larger than a certain value, the theory is in the continuum limit, otherwise the system is in a phase with finite lattice spacing, describing a lattice condensed-matter-like system. In Refs. [16, 17] it is shown there that, when $\langle \overline{\chi}\chi \rangle$ is plotted versus the monopole density ρ_m, data points for QED_3 with $N_f = 0$ and $N_f = 2$ in the compact and noncompact formulations in the lattice regime fall on the same curve to a good approximation (see Fig. 2 (left)). This led the authors of Refs. [16, 17] to conclude that the physics of the chiral symmetry breaking is the same in the two theories. We want to verify, by the same method, if this conclusion can be extended to the continuum limit.

3. Numerical results

Our Monte Carlo simulation code was based on the hybrid updating algorithm, with a microcanonical time step set to $dt = 0.02$. We simulated one flavour of staggered fermions corresponding
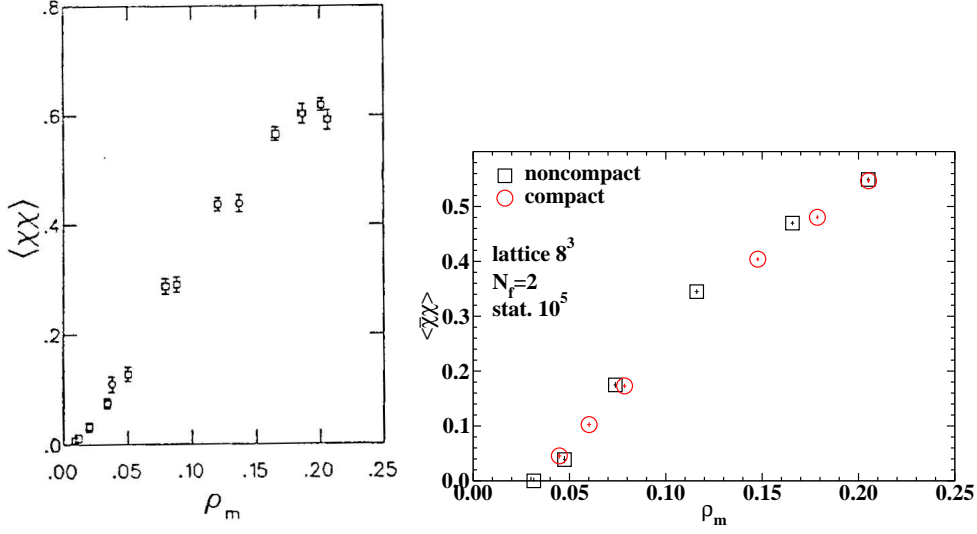
Figure 2: Left: Correlation between $\langle\overline{\chi}\chi\rangle$ and ρ_m for the compact (circles) and the noncompact (boxes) theories for $N_f = 2$ and 8^3 lattice according to Ref. [17] (left) and to our results (right).

to two flavours of 4-component fermions. Most simulations were performed on a 12^3 lattice, for bare quark mass ranging in the interval $am = 0.01 \div 0.05$. We made refreshments of the gauge (pseudofermion) fields every 7 (13) steps of the molecular dynamics. In order to reduce autocorrelation effects, "measurements" were taken every 50 steps. Data were analyzed by the jackknife method combined with binning.

As a first step, we have reproduced the results by Fiebig and Woloshyn which are shown in Fig. 2 (left). We find that also in our case data points from the two formulations nicely overlap (see Fig. 2 (right)).

Then, in Fig. 3 (left) we plot data for $\beta^2\langle\overline{\chi}\chi\rangle$ obtained in the compact formulation versus βm. We restrict our attention to the subset of β values for which data points fall approximately on the same curve (this indicating the onset of the continuum limit) which in the present case means $\beta = 1.9, 2.0, 2.1$, corresponding to $L/\beta = 6.31, 6.00, 5.71$. The ratio L/β fixes the physical volume that is pratically constant in the considered range. A linear fit of these data points gives χ^2/d.o.f. $\simeq 8.4$ and the extrapolated value for $\beta m \to 0$ turns out to be $\beta^2\langle\overline{\chi}\chi\rangle = (1.54 \pm 0.25) \times 10^{-3}$. Restricting the sample to the data at $\beta = 2.1$, the χ^2/d.o.f. lowers to $\simeq 1.3$ and the extrapolated value becomes $\beta^2\langle\overline{\chi}\chi\rangle = (0.94 \pm 0.28) \times 10^{-3}$, thus showing that there is a strong instability in the determination of the chiral limit. If instead a quadratic fit is used for the points obtained with $\beta = 1.9, 2.0, 2.1$, we get $\beta^2\langle\overline{\chi}\chi\rangle = (0.91 \pm 0.45) \times 10^{-3}$ with χ^2/d.o.f. $\simeq 8.7$. Owing to the large uncertainty, this determination turns out to be compatible with both the previous ones.

In Fig. 3 (right) we plot data for $\beta^2\langle\overline{\chi}\chi\rangle$ obtained in the noncompact formulation versus βm. Following the same strategy outlined before, we restrict our analysis to the data obtained with $\beta = 0.7, 0.75, 0.8$, which correspond to $L/\beta = 17.14, 16, 15$. If we consider a linear fit of these data and extrapolate to $\beta m \to 0$, we get $\beta^2\langle\overline{\chi}\chi\rangle = (0.45 \pm 0.03) \times 10^{-3}$ with χ^2/d.o.f. $\simeq 17$. Performing the fit only on the data obtained with $\beta = 0.8$, for which a linear fit gives the best χ^2/d.o.f. value $\simeq 16$, we obtain the extrapolated value $\beta^2\langle\overline{\chi}\chi\rangle = (0.66 \pm 0.07) \times 10^{-3}$. Therefore,

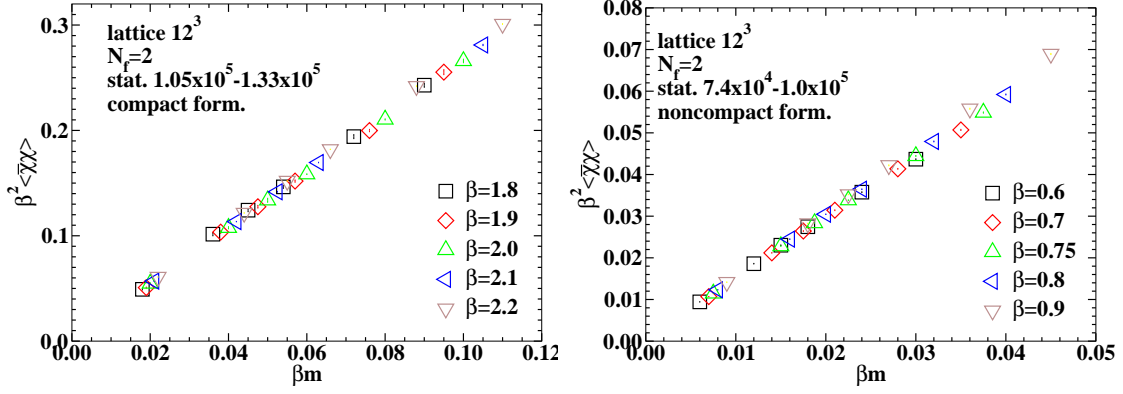**Figure 3:** $\beta^2\langle\overline{\chi}\chi\rangle$ versus βm in the compact (left) and in the noncompact (right) formulation.
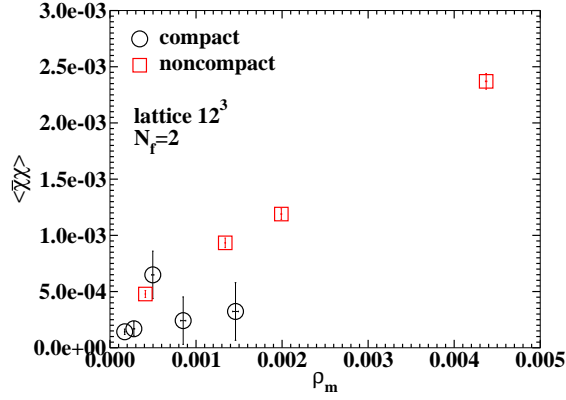

Figure 4: $\langle\overline{\chi}\chi\rangle$ versus ρ_m in both the compact and the noncompact formulations on a 12^3 lattice.

also in the noncompact formulation the chiral extrapolation resulting from a linear fit is largely unstable. A quadratic fit in this case gives instead a negative value for $\beta^2\langle\overline{\chi}\chi\rangle$.

The comparison of the extrapolated value for $\beta^2\langle\overline{\chi}\chi\rangle$ in the two formulations is difficult owing to the instabilities of the fits and to the low reliability of the linear fits, as suggested by the large values of the χ^2/d.o.f. Taking an optimistic point of view, one could say that the extrapolated $\beta^2\langle\overline{\chi}\chi\rangle$ for $\beta = 2.1$ in the compact formulation is compatible with the extrapolated value obtained in the noncompact formulation for $\beta = 0.8$.

It is worth mentioning that our results in the noncompact formulation are consistent with known results: indeed, if we carry out a linear fit of the data for $\beta = 0.6, 0.7, 0.8$ and *am*=0.02, 0.03, 0.04, 0.05 and extrapolate, we get $\beta^2\langle\overline{\chi}\chi\rangle = (1.30 \pm 0.07) \times 10^{-3}$ with an admittedly large χ^2/d.o.f. $\simeq 20$, but very much in agreement with the value $\beta^2\langle\overline{\chi}\chi\rangle = (1.40 \pm 0.16) \times 10^{-3}$ obtained in Ref. [19]. We stress that our results are plagued by strong finite volume effects, therefore our conclusions on the extrapolated values of $\beta^2\langle\overline{\chi}\chi\rangle$ are significant only in the compact versus noncompact comparison we are interested in.

In Fig. 4 we plot $\langle\overline{\chi}\chi\rangle$ versus the monopole density ρ_m. Differently from Fig. 2, it is not evident with the present results that the two formulations are equivalent also in the continuum limit, although such an equivalence cannot yet be excluded. We arrive at the same conclusion

considering a 32^3 lattice.

We studied also $\beta^3 \rho_m$ versus βm for the two formulations (for plots and more details, see Ref. [1]). Our result show that data at different β values do not fall on a single curve, this suggesting that the continuum limit has not been reached for the monopole density for $\beta = 2.2$ in the compact formulation and $\beta = 0.9$ in the noncompact one. We found, however, that the monopole density is independent from the fermion mass. Since the mechanism of confinement in the theory with infinitely massive fermions, i.e. in the pure gauge theory, is based on monopoles and since the monopole density is not affected by the fermion mass, we may conjecture that this same mechanism holds also in the chiral limit.

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