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Topological charge renormalization: A test case for 3-loop vacuum calculations using overlap fermions and Symanzik improved gluons

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We calculate perturbative renormalization properties of the topological charge, using the standard lattice discretization given by a product of twisted plaquettes. We use the overlap and clover action for fermions, and the Symanzik improved gluon action for 4- and 6-link loops.

We compute the multiplicative renormalization of the topological charge density to one loop; this involves only the gluon part of the action. The power divergent additive renormalization of the topological susceptibility is calculated to 3 loops.

XXIIIrd International Symposium on Lattice Field Theory
25-30 July 2005
Trinity College, Dublin, Ireland

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PoS(LAT2005)230

1. Introduction

Topological properties of QCD are among those most widely studied on the lattice. Various methods have been used to this end, involving renormalization, cooling, fermionic zero modes, geometric definitions, etc. In recent years, the advent of fermionic actions, such as the overlap, which do not violate chirality, has brought a new thrust to the subject.

In this work we compute the renormalization constants which are necessary in order to extract topological properties, in the "field theoretic" approach, from Monte Carlo simulations using Wilson or Symanzik improved gluons, and clover or overlap fermions. We compute the multiplicative renormalization $\mathbf{Z}_Q$ of the topological charge density, to 1 loop in perturbation theory, and the power divergent additive renormalization $\mathbf{M}(g^2)$ of the topological susceptibility, to 3 loops.

The main motivations for doing this work are: a) To enable comparison between different approaches used in studying topology, so that a coherent picture of topology in QCD may emerge. b) To enable studies, in numerical simulations, of quantities involving the density of topological charge, $q(x)$, rather than only the integrated charge; this is necessary, e.g., for studying the spin content of nucleons. c) As a feasibility study in lattice perturbation theory: Indeed, this is the first 3-loop calculation to appear in the literature, involving overlap fermions.

A more extensive write-up of this work can be found in our Ref. [1].

2. Computation of $\mathbf{Z}_Q$

Our first task is to compute the multiplicative renormalization Z_Q [2] of the topological charge density $q_L(x)$ to one loop, using the background field method. We use the standard definition of q_L, given by a product of twisted plaquettes

$$q_L(x) = -\frac{1}{2^9\pi^2} \sum_{\mu\nu\rho\sigma=\pm1}^{\pm4} \varepsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\left[\Pi_{\mu\nu}(x)\Pi_{\rho\sigma}(x)\right] \tag{2.1}$$

($\varepsilon_{-\mu,\nu,\rho,\sigma} \equiv -\varepsilon_{\mu,\nu,\rho,\sigma}$; in standard notation: $\Pi_{\mu\nu}(x) = U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x)$.)

The classical limit of the operator shown in Eq.(2.1) must be corrected by including a renormalization function Z_Q, which can be expressed perturbatively as

$$Z_Q = 1 + Z_1 \cdot g^2 + \cdots, \qquad Z_1 = Z_{11} \cdot N_c + Z_{12}/N_c \tag{2.2}$$

We perform a calculation of Z_1; this involves only the gluon part of the action.

In the background field method, link variables are decomposed as

$$U_\mu(x) = V_\mu(x)U_{c\mu}(x), \qquad V_\mu(x) = e^{igQ_\mu(x)}, \qquad U_{c\mu}(x) = e^{iaB_\mu(x)} \tag{2.3}$$

in terms of links for a quantum field and a classical background field, respectively.

The diagrams involved in the one-loop calculation of Z_Q are shown in Figure 1.

The standard Symanzik improved gauge field action, with 4- and 6-link Wilson loops, is

$$\begin{aligned} S_G = \frac{2}{g^2} \Big[& c_0 \sum_{\text{plaquette}} \mathrm{Re}\,\mathrm{Tr}\,(1 - U_{\text{plaquette}}) + c_1 \sum_{\text{rectangle}} \mathrm{Re}\,\mathrm{Tr}\,(1 - U_{\text{rectangle}}) \\ & + c_2 \sum_{\text{chair}} \mathrm{Re}\,\mathrm{Tr}\,(1 - U_{\text{chair}}) + c_3 \sum_{\text{parallelogram}} \mathrm{Re}\,\mathrm{Tr}\,(1 - U_{\text{parallelogram}}) \Big] \end{aligned} \tag{2.4}$$

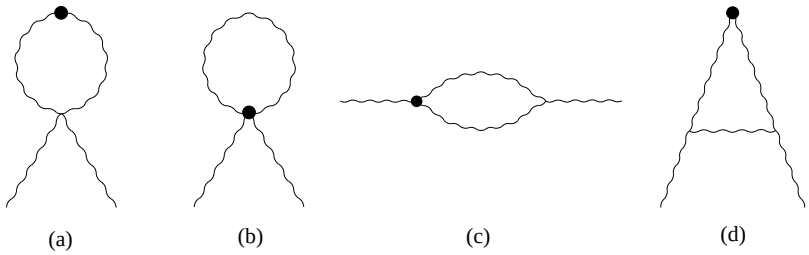


Figure 1: Diagrams contributing to Z_1. The bullet stands for topological charge vertices.

The Symanzik coefficients c_i must satisfy: $c_0 + 8c_1 + 16c_2 + 8c_3 = 1$.

Our calculations are performed without assumptions on the values of the external momenta p_1, p_2 : This is safest for q_L, otherwise one may easily end up with indeterminate expressions; it entails handling 3-point form factors [3] in $D = 4 - 2\varepsilon$ dimensions, such as:

$$\bar{C}_{\mu\nu}(a, p_1, p_2) \;=\; \frac{(ka)^{2\varepsilon}}{a^0} \int \frac{d^D k}{(2\pi)^D} \frac{sink_\mu \, sink_\nu}{\hat{k}^2 \, \widehat{(k+ap_1)}^2 (k + \widehat{ap_1 + ap_2})^2} \tag{2.5}$$

Diagrams (c) and (d) of Figure 1, taken separately, exhibit poles in ε ((d) $\propto -1/\varepsilon - ln\,\kappa^2 a^2$). These cancel, however, upon summation, as is expected by the fact that Q does not renormalize in the continuum. The calculation of Z_Q is particularly involved in the present case, involving propagators and vertices from the improved gluonic action. In particular, the calculation of diagram (d) involves a summation of $> 1\,000\,000$ different algebraic expressions at intermediate stages.

Our results for Z_Q are listed in Table 1. In all calculations that involve the parameters c_i, we choose a standard set of values, as in Ref. [4]. The choice of the sets of parameters correspond to the most popular actions: The first set corresponds to the plaquette action, the second set corresponds to the tree-level Symanzik improved action [5] and the next 6 sets correspond to the tadpole improved Lüscher-Weisz (TILW) action [6] for 6 values of beta: $\beta = 8.60, 8.45, 8.30, 8.20, 8.10, 8.00$. The last two sets correspond to the Iwasaki [7] and DBW2 [8] actions, respectively.

In the case of the plaquette action, our result agrees with the known result of Ref. [2].

It is worth noting that the value of Z_1 (and of e_3, see below) for the DBW2 action is the smallest one, leading to a renormalization factor Z_Q closer to 1 (and $M(g^2)$, Eq. (3.2), closer to 0). This would single out the DBW2 action as a better candidate for studies of topology.

3. Computation of $\mathbf{M}(g^2)$

The second task we attend to is the calculation of the additive renormalization of the topological charge susceptibility, which is defined as

$$\chi_L \;=\; \sum_x \langle q_L(x) q_L(0) \rangle \tag{3.1}$$

χ_L develops an unphysical background term which becomes dominant in the continuum limit

$$\chi_L(g^2) \;=\; a^4 Z_Q(g^2)^2 \chi + M(g^2), \qquad\qquad M(g^2) = e_3 \cdot g^6 + e_4 \cdot g^8 + \cdots \tag{3.2}$$

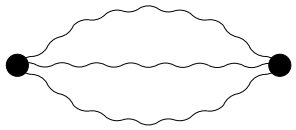
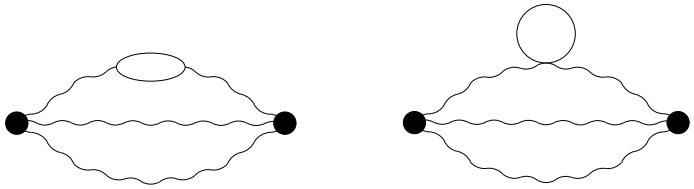


Figure 2: 2-loop diagram contributing to e_3. Bullets stand for topological charge vertices.



(a) (b)

Figure 3: Diagrams contributing to e_4^f. Wavy (straight) lines correspond to gluons (overlap/clover fermions).

$M(g^2)$ is the power divergent additive renormalization of χ_L . We compute the 2-loop coefficient e_3 . This quantity is evaluated for several sets of values of the Symanzik improvement coefficients. Figure 2 shows the diagram contributing to e_3 .

The 3-loop term e_4 of the expansion of $M(g^2)$ equals $e_4 = e_4^g + e_4^f$, where e_4^f stands for the fermionic contribution to e_4 ($c_{\rm SW}$ is the coefficient in the clover action)

$$e_4^f = N_f(N_c^2-1)N_c \cdot (e_{4,0} + e_{4,1}\, c_{\rm SW} + e_{4,2}\, c_{\rm SW}^2) \tag{3.3}$$

and e_4^g is the purely gluonic contribution, expressed as in Ref. [9]

$$e_4^g = \frac{1}{16}(N_c^2-1)(1.735N_c^2 - 10.82 + 73.83/N_c^2)\times 10^{-7} \tag{3.4}$$

In fact, what we are interested in, is the calculation of the parameters $e_{4,0}\,, e_{4,1}\,, e_{4,2}$. This task is performed using both overlap and clover fermions (clearly, overlap fermions involve only the parameter $e_{4,0}$). Figure 3 shows the 3-loop diagrams contributing to the evaluation of e_4^f.

The propagator and vertices of overlap fermions can be obtained from the following expression for the overlap action, written in terms of the massless Neuberger-Dirac operator D_N [10]

$$S_{Overlap} = a^4 \sum_{n,m} \bar{\Psi}(n) D_N(n,m) \Psi(m), \qquad D_N = \frac{M_0}{a}\left(1 + \frac{X}{\sqrt{X^\dagger X}}\right) \tag{3.5}$$

M_0 is a real parameter corresponding to a negative mass term. M_0 must lie in the range $0 < M_0 < 2r$, r being the Wilson perameter (in our case $r = 1$). X is the Wilson-Dirac operator with mass $-M_0$.

The clover (SW) fermionic action [11], contains an extra term, parameterized by a coefficient, $c_{\rm SW}$; this coefficient is treated here as a free parameter.

In performing this calculation, a large effort was devoted to the creation of an efficient 3-loop "integrator", that is, a metacode for converting lengthy 3-loop integrands into efficient code for numerical integration. The output code of the integrator precalculates a number of time-consuming common ingredients (Symanzik propagator, overlap expressions, etc.), exploits symmetries of the

integration region, integrates in parallel over non-overlapping loops, organizes the integrand as an inverse tree for optimized evaluation of innermost loops, etc.

Table 1 contains our results for e_3 (cf. Eq.(3.2)) for different gluonic actions. These results, for the case of the plaquette action, agree with older known results (see, e.g., [12]). Figure 4 shows the coefficients $e_{4,0}$, $e_{4,1}$, $e_{4,2}$ of the clover result for different values of the bare fermion mass m. Figure 5 exhibits the dependence of e_4 , using the overlap action, on the parameter M_0.

A complete tabular version of our results on e_4 , for both the clover and overlap cases, can be found in our longer write-up, Ref. [1].

c_0	c_1	c_3	Z_{11}	Z_{12}	$e_3 \times 10^{-7}$
1.0	0.0	0.0	-0.33059398205(2)	0.2500000000(1)	6.89791329(1)
1.6666666	-0.083333	0.0	-0.2512236240(1)	0.183131339233(1)	3.1814562840(7)
2.3168064	-0.151791	-0.0128098	-0.20828371039(3)	0.147519438874(3)	1.8452250005(2)
2.3460240	-0.154846	-0.0134070	-0.20674100461(1)	0.146259768983(1)	1.8054229585(4)
2.3869776	-0.159128	-0.0142442	-0.20462181183(1)	0.144531861677(4)	1.7516351593(8)
2.4127840	-0.161827	-0.0147710	-0.20331145580(1)	0.143464931830(1)	1.7188880608(5)
2.4465400	-0.165353	-0.0154645	-0.20162651307(1)	0.142094444611(2)	1.6773505020(9)
2.4891712	-0.169805	-0.0163414	-0.19954339172(1)	0.140402610424(1)	1.626880218(1)
3.648	-0.331	0.0	-0.15392854668(1)	0.105132852383(2)	0.752432061(7)
12.2688	-1.4086	0.0	-0.0617777059(4)	0.038277296152(6)	0.04881939(4)

Table 1: The values of Z_{11} and Z_{12} (Eq.(2.2), Figure 1), and of e_3 (Eq.(3.2), Figure 2), with Symanzik improved gluons, for various values of the coeffi cients c_0, c_1, c_3 . $(c_2 = 0)$

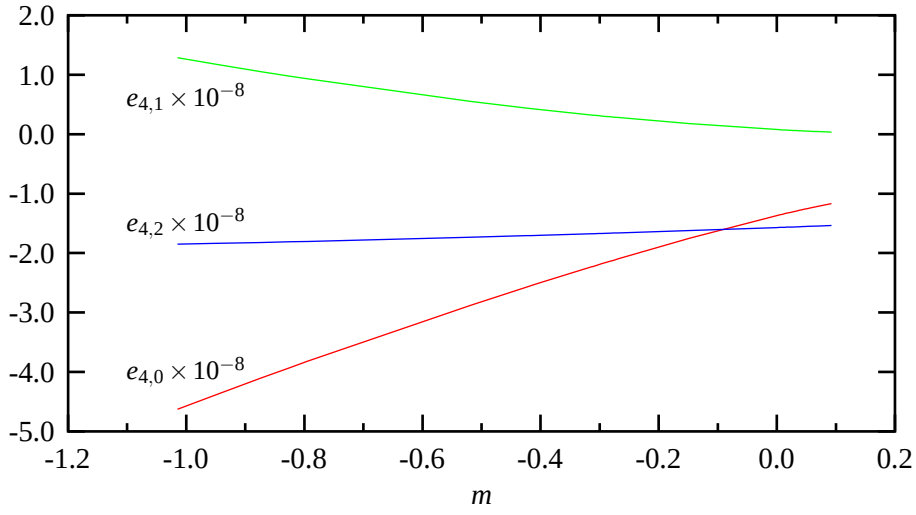


Figure 4: Variation of the terms contributing to e_4^f as a function of m

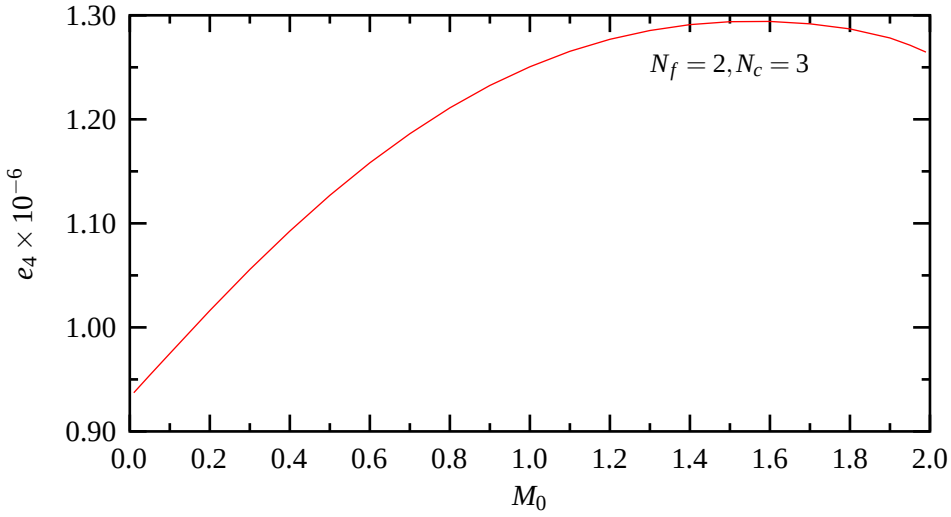


Figure 5: Value of e_4 as a function M_0

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