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CKM matrix with lattice QCD
– the full determination using recent results –

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A full determination of the CKM matrix using recent results from lattice QCD is presented. To extract the CKM matrix in a uniform fashion, I exclusively use results from unquenched lattice QCD as the theory input for nonperturvative QCD effects. All 9 CKM matrix elements and all 4 Wolfenstein parameters are obtained from results for gold-plated quantities, which include semileptonic decay form factors and leptonic decay constants of B, D and K mesons, and $B^0-\bar{B}^0$ and $K^0-\bar{K}^0$ mixing amplitudes.

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1. Introduction

The Cabibbo-Kobayashi-Maskawa (CKM) matrix [1, 2] is a set of fundamental parameters in the Standard Model, which relates the mass eigenstates and the weak eigenstates in the electroweak theory. In the Wolfenstein parameterization [3], it is given by

$$V_{\rm CKM} = \begin{pmatrix} \mathbf{V_{ud}} & \mathbf{V_{us}} & \mathbf{V_{ub}} \\ 1-\lambda^2/2 & \lambda & A\lambda^3(\rho-i\eta) \\ \mathbf{V_{cd}} & \mathbf{V_{cs}} & \mathbf{V_{cb}} \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ \mathbf{V_{td}} & \mathbf{V_{ts}} & \mathbf{V_{tb}} \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} \tag{1.1}$$

Because of unitarity, it contains only 4 independent parameters, (λ, A, ρ, η).

To determine each CKM matrix element, one requires both theoretical and experimental inputs. On the theoretical side, one needs to know relevant hadronic amplitudes, which often contain nonperturbative QCD effects. A major role of lattice QCD is to calculate such hadronic amplitudes nonperturbatively, from first principles. One can then extract the CKM matrix elements by combining lattice QCD results as the theoretical input with the experimental input such as branching fractions.

To accurately determine each CKM matrix element from lattice QCD, one should use hadronic processes whose amplitude can be reliably calculated with existent technique. There are a set of such processes (amplitudes) — so-called "gold-plated" processes (quantities) which contain at most one hadron in initial and final states [4]. These include the exclusive semileptonic decays and leptonic decays of B, D and K mesons, and neutral $B-\bar{B}$ and $K-\bar{K}$ mixings. For such processes, technique for lattice simulations are already well established, thus reliable calculations are possible.

$$\begin{pmatrix} \mathbf{V_{ud}} & \mathbf{V_{us}} & \mathbf{V_{ub}} \\ \pi\to l\nu & K\to\pi l\nu & B\to\pi l\nu \\ & K\to l\nu & \\ \mathbf{V_{cd}} & \mathbf{V_{cs}} & \mathbf{V_{cb}} \\ D\to\pi l\nu & D\to K l\nu & B\to D^{(*)}l\nu \\ D\to l\nu & D_s\to l\nu & \\ \mathbf{V_{td}} & \mathbf{V_{ts}} & \mathbf{V_{tb}} \\ \langle B_d|\overline{B}_d\rangle & \langle B_s|\overline{B}_s\rangle & \end{pmatrix}$$

Figure 1: Gold-plated processes for each CKM matrix element [4]. The neutral $K-\bar{K}$ mixing (characterized by the CP-violating parameter ε_K) is another gold-plated processe, which gives a constraint on the phase of the CKM matrix (ρ, η).

The gold-plated processes for each CKM matrix element are summarized in Fig. 1. The magnitude of the CKM matrix element, *e.g.*, $|V_{cd}|$ can be determined from either the semileptonic decay $D\to\pi l\nu$ or leptonic decay $D\to l\nu$, as explained below. $|V_{ub}|$ and $|V_{td}|$ can be respectively extracted from the semileptonic $B \to \pi l \nu$ decay and neutral $B-\bar{B}$ mixing, which give constraints on the phase of the CKM matrix (ρ,η). The neutral $K-\bar{K}$ mixing give another constraint. Taking these together, one can extract (ρ,η) assuming that the Standard Model is correct. In this way one can, in principle, determine all 9 CKM matrix elements, *i.e.*, all 4 Wolfenstein parameters using lattice QCD.

The accuracy of the CKM matrix elements from lattice QCD is subject to several systematic uncertainties, however. The most serious one is the uncertainty from the "quenched" approximation, in which effect of virtual quark loops (dynamical quarks) is neglected ("$n_f = 0$"). This approximation has been adapted in the community long time, simply to reduce the computational cost. Recent developments of computer resources and algorithms enable us to perform more realistic lattice calculations — "unquenched" simulations which include effect of light (up, down) dynamical quarks ("$n_f = 2$") or light and strange dynamical quarks ("$n_f = 2+1$"). For recent status of the unquenched simulations, see Ref. [5].

Most of gold-plated quantities listed above have been or being calculated in unquenched lattice QCD. Given this situation, it might be a good time to present the CKM matrix elements from lattice QCD in a uniform fashion in one place. This paper gives a result for the whole CKM matrix – all 9 CKM matrix elements and all 4 Wolfenstein parameters – determined from lattice QCD using recent results for gold-plated quantities. For this purpose I exclude results from quenched QCD. However, I note that quenched calculations are still important and useful to study methodology and other systematic uncertainties. For recent reviews on the quenched calculations of gold-plated quantities, see Refs. [6, 7, 8, 9, 10].

Although there are several ways to discretize quarks on the lattice, some of gold-plated quantities have been calculated in unquenched QCD only using the staggered-type fermion[1] so far. This is because the staggered fermion is computationally much faster than other lattice fermion formalisms such as the Wilson-type fermion, domain wall fermion and overlap fermion. As a consequence, the results for the CKM matrix elements in this paper are often estimated from only one or two unquenched calculations. I hope that more unquenched results using other lattice fermion formalisms will come out in the near future, and leave future reviewers to make a more serious average of the CKM matrix.

To present the result for the CKM matrix in a uniform fashion, I use lattice QCD results only as the theory input for nonperturbative QCD effects. It is of course desirable to include non-lattice theory inputs (such as ones for inclusive B decays) to improve the precision, but it is beyond the scope of this paper. For recent progress on non-lattice approaches for CKM phenomenology, see, *e.g.*, Refs. [12, 13, 14].

[1] In the staggered fermion formalism, one needs to take the fourth-root of the fermion determinant to adjust the number of quark flavor, as a consequence of the fermion doubling. The validity of this procedure is not yet proven, so further study on this isuue is necessary. For a review, see Ref. [11].

The results for the magnitudes of the CKM matrix elements are

$$V_{\rm CKM}^{\rm Lat05} = \begin{pmatrix} |\mathbf{V_{ud}}| & |\mathbf{V_{us}}| & |\mathbf{V_{ub}}| \\ 0.9745(3) & 0.2244(14) & 3.76(68)\times10^{-3} \\ |\mathbf{V_{cd}}| & |\mathbf{V_{cs}}| & |\mathbf{V_{cb}}| \\ 0.245(22) & 0.97(10) & 3.91(35)\times10^{-2} \\ |\mathbf{V_{td}}| & |\mathbf{V_{ts}}| & |\mathbf{V_{tb}}| \\ 7.40(79)\times10^{-3} & 3.79(53)\times10^{-2} & 0.9992(1) \end{pmatrix}. \tag{1.2}$$

The results for the Wolfenstein parameters are

$$\lambda = 0.2244(14) \quad , \tag{1.3}$$

$$A = 0.78(7) \ , \tag{1.4}$$

$$\rho = 0.16(7) \ , \tag{1.5}$$

$$\eta = 0.37(4). \tag{1.6}$$

The rest of this paper is organized as follows. In Section 2, I review recent results for the semileptonic and letonic decays of B, D and K mesons, and determine the magnitude of the CKM matrix elements using them. 5 CKM matrix elements are directly determined using lattice results, whereas other elements (except for $|V_{td}|$) are indirectly determined using CKM unitarity. In Section 3, I review recent results for the $B^0-\bar{B}^0$ and $K^0-\bar{K}^0$ mixing amplitudes, and extract the CKM phase (ρ,η) by performing the unitarity triangle analysis. In Section 4, I give the conclusion.

The first attempt of the full determination of the CKM matrix using lattice QCD were made in Ref. [15], where results from semileptonic decays only were used. The determination of the CKM matrix elements using lattice QCD has been discussed long time; see, *e.g.*, Refs. [9, 10] for recent reviews. A more detailed review on recent lattice calculations of K meson physics (such as the $K^0-\bar{K}^0$ mixing and $K\to\pi l\nu$ decay) can be found in Ref. [16].

2. CKM magnitude from lattice QCD

I start this section with examples of the determination of the magnitude of the CKM matrix elements. The CKM magnitude is often determined from the semileptonic decays (such as $D\to\pi l\nu$ and $K\to\pi l\nu$), and in some cases, leptonic decays (such as $D\to l\nu$ and $K\to l\nu$) also provide an independent determination with a comparable precision. Below I take $|V_{cd}|$ as an example to explain how to extract the CKM magnitude.

The branching fraction of the semileptonic decay $D\to\pi l\nu$ is given by

$$\mathrm{Br}(D\to\pi l\nu) \ = \ |V_{cd}|^2\int_0^{q^2_{\rm max}} dq^2 \ f_+(q^2)^2\times(\text{known factor}), \tag{2.1}$$

where $q=p_D-p_\pi$ is the momentum transfer, $q_{\rm max}=(m_D-m_\pi)^2$ and f_+ is the form factor defined below. The relevant hadronic amplitude is conventionally parametrized as

$$\langle\pi(p_\pi)|V^\mu|D(p_D)\rangle \ = \ f_+(q^2)\left[p_D+p_\pi-\frac{m_D^2-m_\pi^2}{q^2}\,q\right]^\mu + \ f_0(q^2)\,\frac{m_D^2-m_\pi^2}{q^2}\,q^\mu,$$

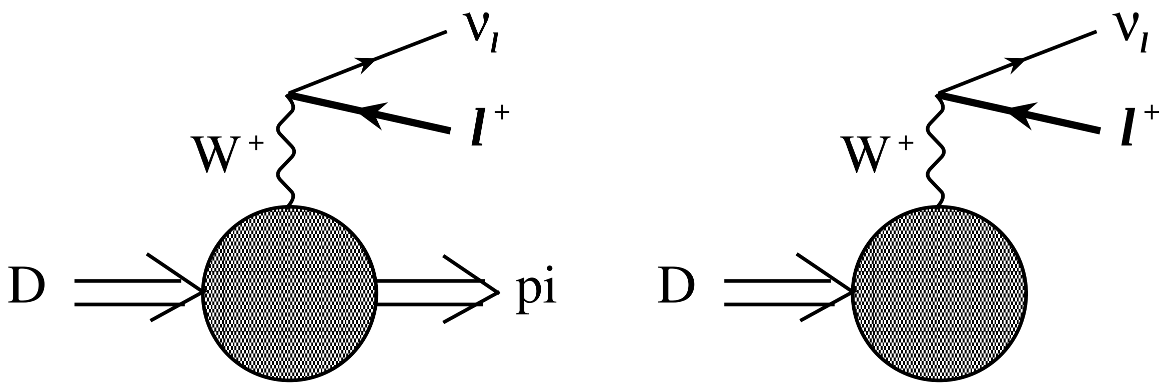

Figure 2: D meson decays relevant to the CLM matrix element $|V_{cd}|$. The left panel is the semileptonic decay $D \to \pi l \nu$, and the right is the leptonic decay $D \to l \nu$.

where V^μ is the vector current involving the heavy and light quarks. By combining the lattice calculation of $f_+(q^2)$ with the experimental measurement of the branching fraction, one can extract $|V_{cd}|$. Similarly $|V_{cs}|$, $|V_{cb}|$, $|V_{ub}|$, and $|V_{us}|$ can be respectively extracted from $D \to K l \nu$, $B \to D l \nu$, $B \to \pi l \nu$ and $K \to \pi l \nu$, as listed in Fig. 1.

The branching fraction of the leptonic decay $D \to l \nu$ is given by

$$\mathrm{Br}(D \to l\nu) \;=\; (\text{known factor}) \times {f_D}^2 |V_{cd}|^2 \tag{2.2}$$

where f_D is the D meson decay constant defined through

$$\langle 0|A^\mu|D(p)\rangle \;=\; f_D\, p^\mu, \tag{2.3}$$

where A^μ is the axial vector current involving the heavy and light quarks. Lattice results for f_D can be used to determine $|V_{cd}|$ once the branching fraction is measured by experiment. The branching fraction of $D \to l\nu$ ($D_s \to l\nu$) decay is being (will be) measured by CLEO-c [17, 18] and that of $K \to l\nu$ have been measured by many groups [19]. Thus they can be used to extract $|V_{cd}|$ ($|V_{cs}|$) and $|V_{us}|$ respectively. On the other hand, $B \to l\nu$ decay is still difficult to measure due to the helicity suppression.

2.1 $|V_{cd}|$ and $|V_{cs}|$ from D meson decays

There are (at least) two reasons to study the D meson decays in lattice QCD. One is to extract the CKM matrix elements, as mentioned above. The other is that, taken the CKM matrix elements from elsewhere, one can test lattice QCD by comparing lattice results for the form factor $f_+^{D\to\pi(K)}$ and decay constant f_D with corresponding experimental results. In particular, if lattice results agree with experiment for the D meson physics, one can have confident in similar quantities for the B meson physics such as $f_+^{B\to\pi}$ and f_B, which are phenomenologically important and cannot be obtained by other means.

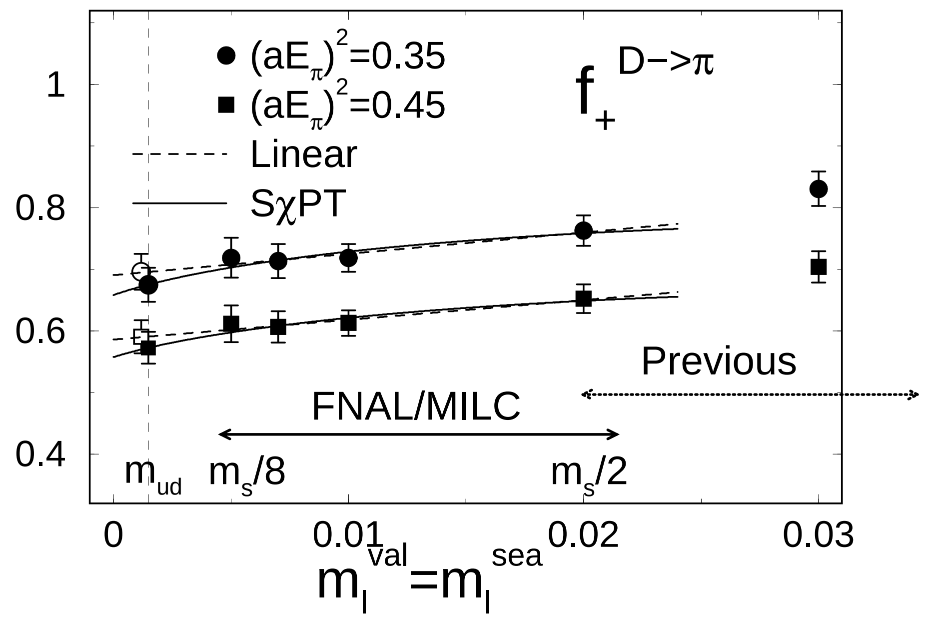

Figure 3: Light quark mass dependence of the $D \to \pi$ form factor in Ref. [20]. Symbols are data points, and lines are chiral fits using the staggered chiral perturbation theory (solid lines) and ones using a linear ansatz (dashed lines). The dashed vertical line indicates the physical ud quark mass.

The Fermilab/MILC collaboration presented the first unquenched ($n_f = 2+1$) calculation of form factors of $D \to \pi l \nu$ and $D \to K l \nu$ decays [20, 21] using the MILC "coarse lattice" ensemble ($a^{-1} \approx 1.6$ GeV) [22]. We use an improved staggered fermion action for light (u,d,s) quarks and an improved Wilson fermion action with the Fermilab interpretation [23] for the charm quark. To combine the staggered light quark (1-component spinor) with the Wilson-type heavy quark, we convert the staggered quark propagator into the naive quark (4-component spinor) propagator according to

$$\Omega(x)^\dagger \langle \bar{\chi}(x)\chi(y)\rangle \Omega(y) = \langle \bar{\psi}(x)\psi(y)\rangle \tag{2.4}$$

where $\Omega(x) = \gamma_0^{x_0}\gamma_1^{x_1}\gamma_2^{x_2}\gamma_3^{x_3}$, as proposed in Refs. [24, 25]. Although the naive quark propagator describes 16 (=2^4) equivalent fermions, known as the "doubling problem", only the physical mode contributes to the low energy physics and other 15 ($= 16-1$) modes decouple when it is combined with the Wilson-type (or any doubler-free) heavy quark. This can be understood by noting that the heavy-light meson mass for each mode is roughly given by $M_D \simeq \{m_u + m_c,\ m_u + (m_c + 2r/a),\ m_u + (m_c + 4r/a), \cdots\}$ with r being the Wilson parameter for the heavy quark action. By taking the asymptotic limit in time direction ($t \to \infty$) for heavy-light 2-point and 3-point functions, the state with the lowest mass (physical mode) can be isolated. This method has been successfully applied to the heavy-light meson physics by the HPQCD collaboration and Fermilab/MILC collaboration.[2]

[2] I note that this method may be applied to the K meson physics as well, with the staggered u,d quarks and the

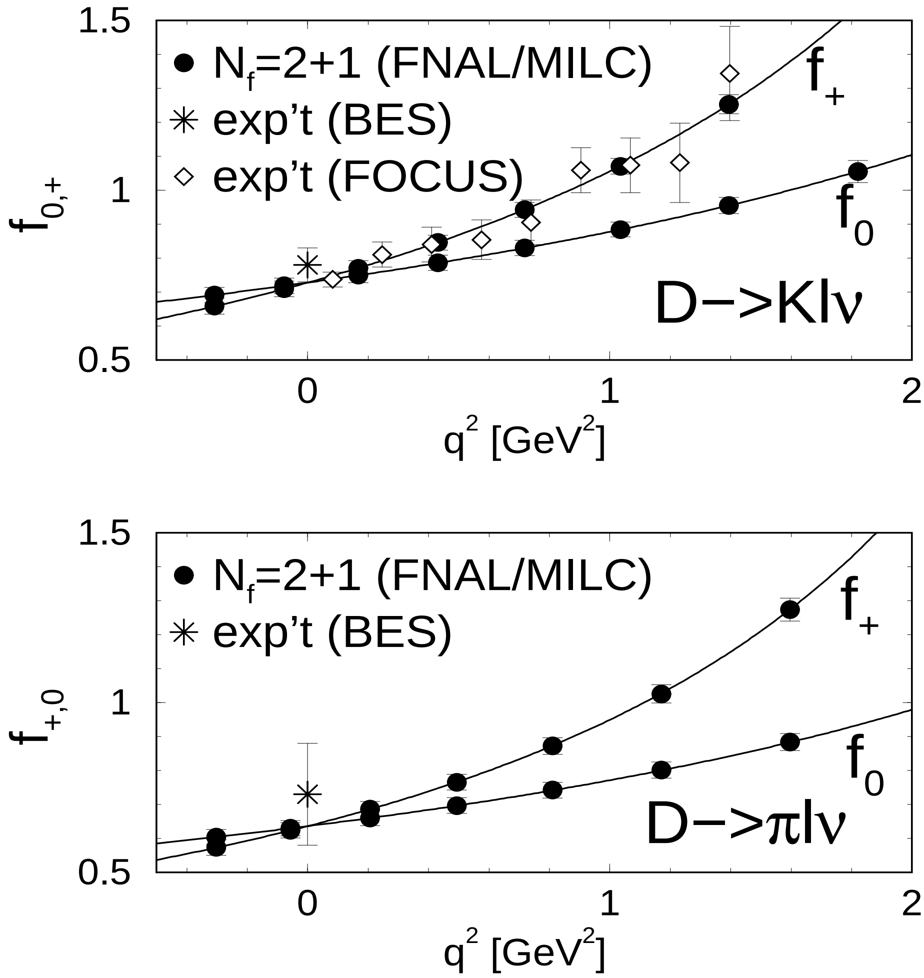

Figure 4: Form factors of $D \to \pi l \nu$ and $D \to K l \nu$ decays in $n_f = 2+1$ lattice QCD by the Fermilab/MILC collaboration [20]. The errors are statistical only. Recent experimental results by the BES collaboration [26] and by the FOCUS collaboration [28] are also shown.

Since the staggered fermion is fast and free from the exceptional configurations, one can perform simulations with the light quark mass m_l as low as $m_s/8$, in contrast to previous calculations using other fermions. This situation is shown in Fig. 3. Physical results at $m_l = m_{ud} \equiv (m_u + m_d)/2$ is obtained from chiral fits using the staggered chiral perturbation theory (SχPT), but a simple linear fit give consistent results. This suggests that the results at the physical light quark mass are insensitive to the fit ansatz and the error from the chiral extrapolation ($m_l \to m_{ud}$) is under control. The key is the availability of data at small light quark masses. On the other hand, since our calculation is done at a single lattice spacing, the error from the lattice discretization effects is still large, giving about 10% total systematic uncertainties for the form factors.

Our final results are shown in Fig. 4 together with recent experimental results [26, 27, 28, 29].

Wilson-type s quark.

The results are obtained using the parameterization of Becirevic and Kaidalov (BK) [30],

$$f_+(q^2) = \frac{F}{(1-\tilde{q}^2)(1-\alpha\tilde{q}^2)}, \quad f_0(q^2) = \frac{F}{1-\tilde{q}^2/\beta}, \tag{2.5}$$

where $\tilde{q}^2 = q^2/m_{D^*}^2$. We obtain [20, 21]

$$F^{D\to\pi} = 0.64(3)(6), \quad \alpha^{D\to\pi} = 0.44(4)(7), \quad \beta^{D\to\pi} = 1.41(6)(13), \tag{2.6}$$

$$F^{D\to K} = 0.73(3)(7), \quad \alpha^{D\to K} = 0.50(4)(7), \quad \beta^{D\to K} = 1.31(7)(13), \tag{2.7}$$

where the first errors are statistical and the second systematic. The results agree with experiment for both the normalization at $q^2 = 0$ [26, 27] and the q^2-dependence [28, 29]. This may indicate reliability of lattice results for the similar quantities for B physics, the $B \to \pi l \nu$ form factor. By integrating out $f_+(q^2)$ in terms of q^2 and using the experimental measured branching fraction [19], we obtain

$$|V_{cd}|_{\rm semi-lep} = 0.239(10)(24)(20), \tag{2.8}$$

$$|V_{cs}|_{\rm semi-lep} = 0.969(39)(94)(24), \tag{2.9}$$

where the first error is statistical, the second systematic, and the third is the experimental error from the branching fractions.

The Fermilab/MILC collaboration has also finalized the $n_f = 2+1$ calculation of the D and D_s meson decay constants [31, 32, 33]. Lattice actions used are the same as ones for the semileptonic calculations. We performed partially quenched simulations, where the valence light quark mass can be different from the dynamical light quark mass, at three values of lattice spacing. The chiral extrapolation is done using the SχPT, and the final results are obtained by taking $a^2\delta \to 0$ after the chiral extrapolation (upper figure of Fig. 5), where $a^2\delta$ denote constants in the SχPT which parametrizes lattice discretization effects from the staggered fermion. After $a^2\delta \to 0$, the lattice spacing dependence is small (lower figure of Fig. 5). Our final results are $f_D = 201(03)(17)$ MeV and $f_{D_s} = 249(03)(16)$ MeV, where the first errors are statistical and the second systematic. Two largest sources of the systematic errors are the discretization effects and the chiral extrapolation (for f_D).

The CP-PACS collaboration reported a new unquenched ($n_f = 2$) calculation of the $D_{(s)}$ decay constant [34, 35]. For the light quarks, they used an $O(a)$ improved Wilson action. For the charm quark, they used a relativistic on-shell improved action which is similar to one in Ref. [23] but derived from a different point of view in Ref. [36]. Their simulated light quark mass ranges $m_l \le m_s/2$, and a linear chiral extrapolation was made. They studied the lattice spacing dependence using three lattice spacings as in the Fermilab/MILC calculation, and performed a continuum extrapolation using two data sets (one is from the temporal axial vector current and the other is from the spatial current) as shown in Fig. 6. Their preliminary results are $f_D = 202(12)(^{+20}_{-25})$ MeV and $f_{D_s} = 238(11)(^{+07}_{-27})$ MeV, where the systematic errors (second ones) are dominated by uncertainties from the continuum extrapolation.

The chirally improved actions were applied to the charm quark physics in two recent quenched ($n_f = 0$) studies. Ref. [37] used the overlap fermion action, whereas Ref. [38] used a variant of the domain-wall fermion action. The use of the chirally improved actions has an advantage that they

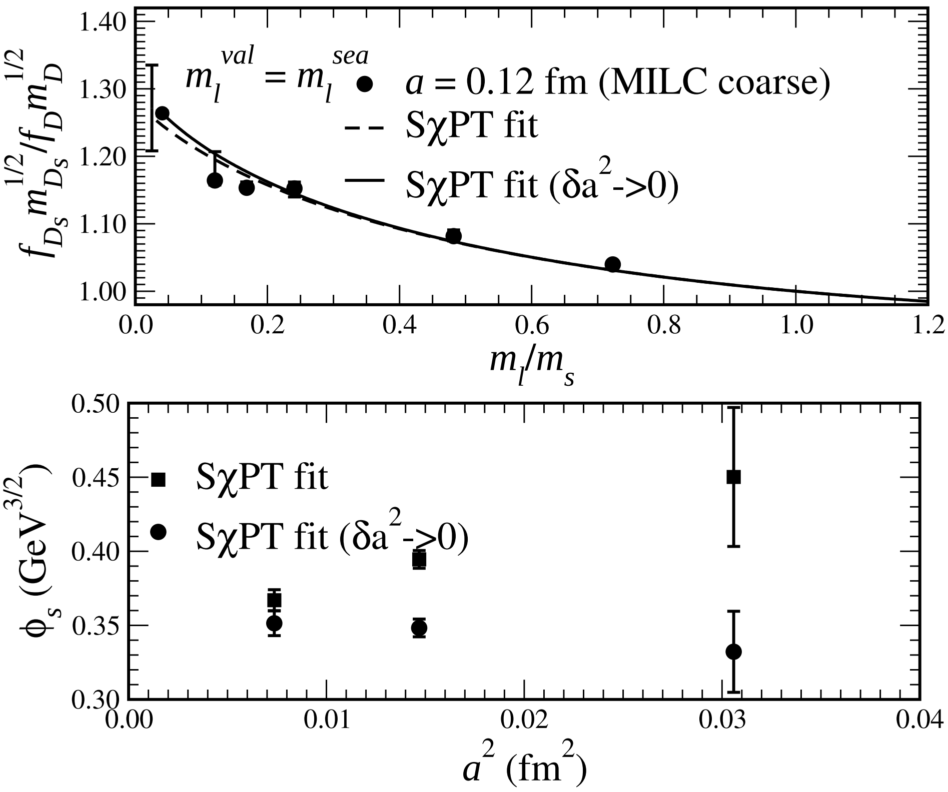

Figure 5: D and D_s meson decay constants in $n_f = 2+1$ lattice QCD by the Fermilab/MILC collaboration [31]. The upper figure shows the light quark mass dependence and chiral extrapolation for $f_{D_s}\sqrt{m_{D_s}}/(f_D\sqrt{m_D})$. The lower figure shows the lattice spacing dependence of $\phi_s = f_{D_s}\sqrt{m_{D_s}}$.

are free from $O(a, m_Q a)$ discretization errors without any tuning of parameters, so may be useful for the simulations involving the charm quark where $m_Q a < 1$.

Turning to the experimental result for the leptonic decay, the CLEO-c collaboration updated their measurement of $D \to \mu\nu$ branching fraction [17, 18]. Assuming that $|V_{cd}| = |V_{us}| = 0.224(3)$, they obtain the experimental result $f_D^{\rm CLEO-c} = 223(17)(03)$ MeV, which is in agreement with the recent lattice results. Their precision, $O(10\%)$, is similar to ones for the lattice results. The agreement may indicate reliability of lattice calculations of the heavy-light decay constants. They will further improve the precision of f_D and report the result for f_{D_s} in the future.

One may conversely use the CLEO-c result for the $D \to \mu\nu$ branching fraction to determine $|V_{cd}|$. If it is combined with the $n_f = 2+1$ lattice result for f_D by the Fermilab/MILC collaboration, one obtains

$$|V_{cd}|_{\rm lept} = 0.250(22)(21), \tag{2.10}$$

where the first error is one from the lattice calculation and the second is from the experimental uncertainty. $|V_{cd}|$ from the leptonic decay is consistent with one from the semileptonic decay Eq. (2.8), and the size of uncertainties is similar to each other.

It is interesting to consider the ratio of the leptonic and semileptonic decay branching fractions

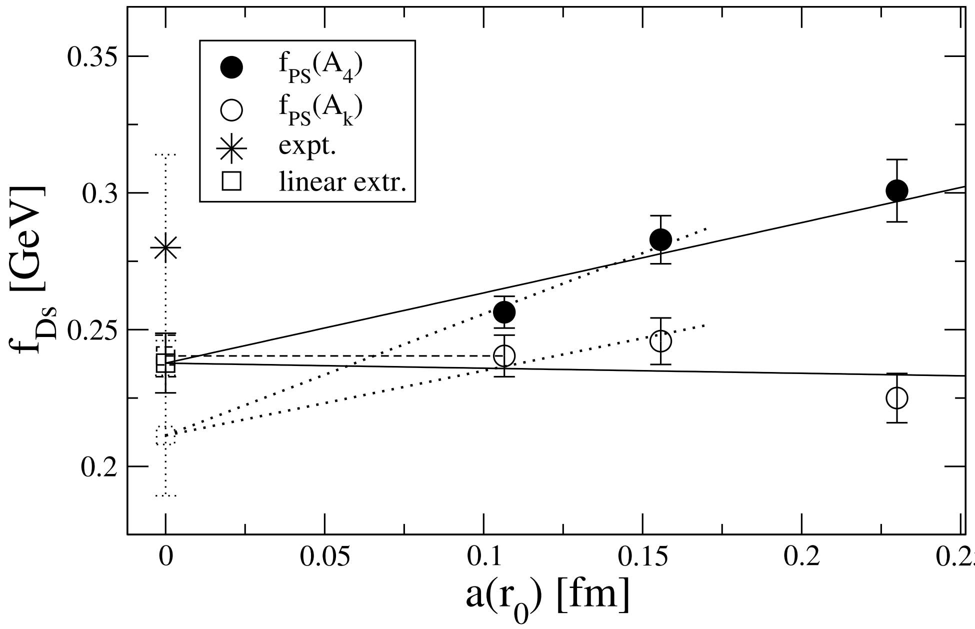

Figure 6: Lattice spacing dependence and continuum extrapolation ($a \to 0$) for f_{D_s} in $n_f = 2$ lattice QCD by the CP-PACS collaboration [34, 35]. Filled circles are results from the temporal axial vector current and open circles are from the spatial one.

because the CKM matrix element $|V_{cd}|$ is canceled in the ratio. Writing

$$R \equiv \sqrt{\frac{\mathrm{Br}(D \to l\nu)}{\mathrm{Br}(D \to \pi l\nu)}}, \tag{2.11}$$

the R is proportional to $f_D/[\int dq^2 (f_+^{D\to\pi}(q^2))^2]^{1/2}$. Since one can directly compare lattice results with experiment for this quantity, it provides a good test of lattice QCD. The experimental result (dominated by the CLEO-c measurements) is [18]

$$R_{\mathrm{exp}} = 0.25(2), \tag{2.12}$$

and the unquenched lattice result using the Fermilab/MILC calculation of f_D and $f_+^{D\to\pi}$ is

$$R_{\mathrm{lat}} = 0.22(2). \tag{2.13}$$

They are in reasonable agreement with each other.

As for the Lattice'05 value of the CKM matrix elements, I take an weighted average of Eqs. (2.8) and (2.10) for $|V_{cd}|$, and simply quote Eq. (2.9) for $|V_{cs}|$;

$$|V_{cd}|_{\mathrm{Lat05}} = 0.245(22) \ , \tag{2.14}$$

$$|V_{cs}|_{\mathrm{Lat05}} = 0.97(10) \ , \tag{2.15}$$

where the errors are the combined uncertainties from theory (from lattice QCD) and experiment. These are consistent with the values quoted in Particle Data Group (PDG) [19]; $|V_{cd}|_{\mathrm{PDG}} = 0.224(12)$

and $|V_{cs}|_{\rm PDG} = 0.996(13)$. I note that the PDG values above are obtained from neither the semileptonic nor the leptonic decays, thus Eqs. (2.14) and (2.15) provide an independent determination of $|V_{cd}|$ and $|V_{cs}|$.

2.2 B meson decays

2.2.1 $B \to \pi l \nu$ decay

The semileptonic decay $B \to \pi l \nu$ can be used to determine $|V_{ub}|$, which is one of the most important CKM parameters to constraint the unitarity triangle. Since the pion momentum available in lattice calculations are limited up to around 1 GeV, only higher q^2-region ($q^2 \geq 16$ GeV^2) can be simulated for the $B \to \pi l \nu$ decay. This is in contract to the $D \to \pi l \nu$ decay, for which we can cover all q^2-region ($0 \leq q^2 \leq q^2_{\rm max}$). Since the experimental accuracy of the branching fraction is better for a lower q^2-region than a higher q^2-region, it is desirable to extend the lattice result to the lower q^2-region. Below I first summarize recent lattice results for the $B \to \pi l \nu$ form factors, and extract $|V_{ub}|$ using the results for the higher q^2-region. I then discuss recent attempts to extend the results to the lower q^2-region.

There are two $n_f = 2+1$ unquenched calculations of the $B \to \pi l \nu$ form factors using the MILC coarse lattice ensembles with improved staggered light quarks. One is obtained with the NRQCD heavy quark [39] by the HPQCD collaboration [40, 41], and the other is with the Fermilab heavy quark by the Fermilab/MILC collaboration [21, 42]. Their results are shown in Fig. 7. The total systematic uncertainties are 11% for the form factors in both calculations. The largest quoted error comes from the perturbative matching between the lattice and continuum theory for the HPQCD result, and from the discretization effect for the Fermilab/MILC result. Two unquenched results agree with each other within errors. These are also consistent with previous quenched calculations [43, 44, 45, 46, 47]; at present it is difficult to estimate the effect of the quenching quantitatively because the size of other systematic uncertainties $O(10\%)$ is the same as the expected size of the quenching error.

By integrating $f_+(q^2)$ over 16 GeV$^2 \leq q^2 \leq q^2_{\rm max}$ and using the partial branching fraction Br(16 GeV$^2 \leq q^2 \leq q^2_{\rm max}$) measured by CLEO [48], Belle [49, 50] and Babar [51, 52], the CKM matrix element is obtained as

$$|V_{ub}|_{\rm HPQCD} = 4.04\ (20)(44)(53)\times 10^{-3}, \tag{2.16}$$

$$|V_{ub}|_{\rm FNAL/MILC} = 3.48\ (29)(38)(47)\times 10^{-3}, \tag{2.17}$$

where the first error is statistical, the second systematic, and the third is the experimental error. As for the Lattice 2005 value, I take a simple average of the two results, obtaining

$$|V_{ub}|_{\rm Lat05} = 3.76(68)\times 10^{-3}. \tag{2.18}$$

The total uncertainty for $|V_{ub}|$ is 18%.

Let us discuss the q^2-dependence of the $B \to \pi l \nu$ form factors. Both HPQCD and Fermilab/MILC collaborations use the BK parameterization Eq. (2.5) to interpolate and extrapolate the results in q^2. To estimate the uncertainty from the BK parameterization, the Fermilab/MILC collaboration also made a fit using a polynomial ansatz in q^2. The difference between the two methods

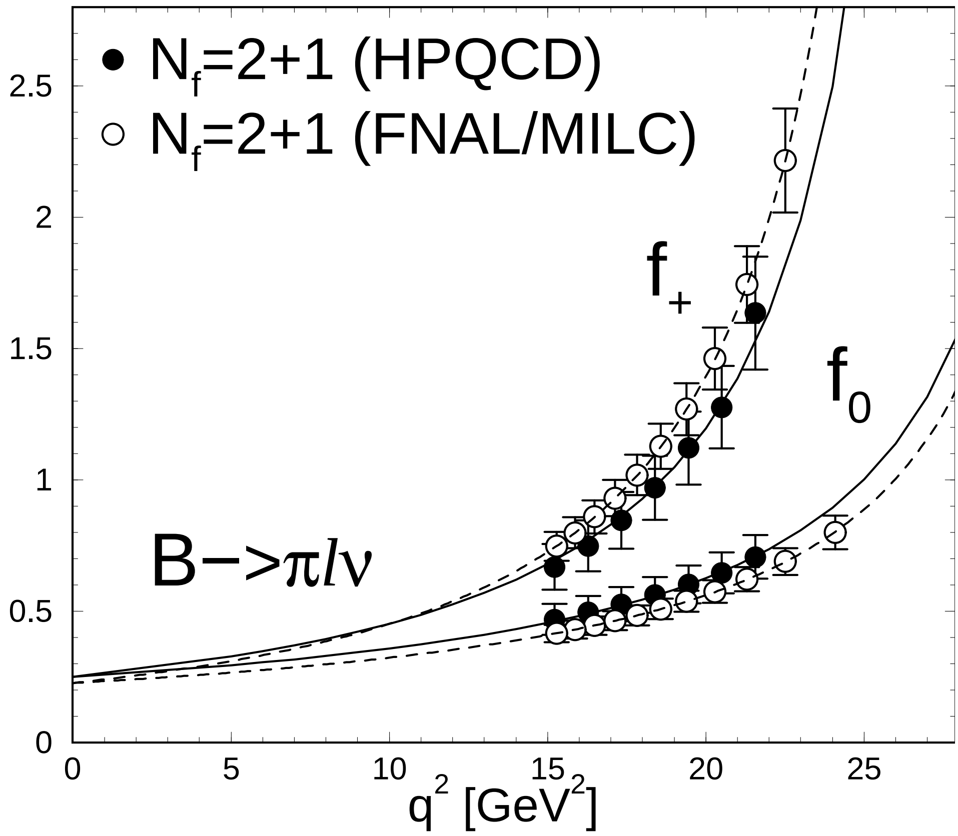

Figure 7: Form factors of the $B \to \pi l \nu$ decay in $n_f = 2+1$ lattice QCD by the HPQCD collaboration [40, 41] and by the Fermilab/MILC collaboration [21, 42]. Lines are ones from fits with the BK parameterization.

for $|V_{ub}| \propto (\int_{q^2_{\rm min}}^{q^2_{\rm max}} dq^2 f_+(q^2)^2)^{-1/2}$ is 4% with $q^2_{\rm min} = 16$ GeV^2. This difference is included in the systematic error in Eq. (2.17). With $q^2_{\rm min} = 0$, however, the difference amounts to be 11%, which is a significant effect. This is because a long extrapolation is required from lattice data points (16 GeV$^2 \le q^2 \le q^2_{\rm max}$) to $q^2 = 0$. For a more precise determination of $|V_{ub}|$, it is necessary to reduce the uncertainty for the lower q^2-region.

One solution is to combine the lattice results for the higher q^2-region with non-lattice results for the lower q^2-region. Ref. [53] combined the recent unquenched lattice results with the QCD dispersion relation and $|V_{ub}| f_+(0)$ from an analysis of the $B \to \pi\pi$ decay based on the Soft Collinear Effective Theory (SCET). The QCD dispersion relation is not a model but an analyticity bound. Ref. [54] made a similar study using $f_+(0)$ from the light-cone sum rule instead of one from the SCET. Ref. [55] combined quenched lattice results with the QCD dispersion relation and the experimental measured q^2-dependence. In each case, the accuracy of $|V_{ub}|$ can be improved with the additional informations on the form factors; it is 11–14%, better than one from lattice results alone (18%); *c.f,* Eq. (2.18).

Another solution is a direct lattice simulation at lower q^2 using the moving NRQCD (mN-

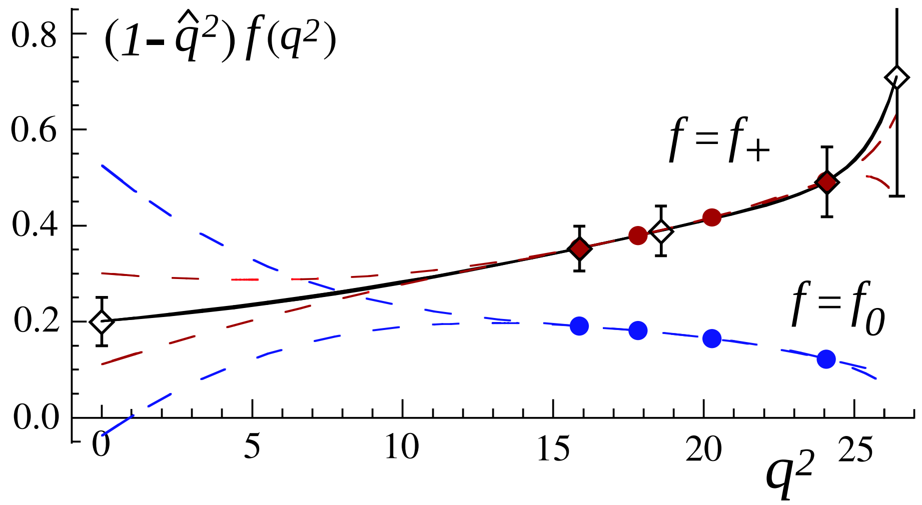

Figure 8: $B \to \pi l \nu$ form factors using the unquenched lattice results, QCD dispersion relation and $|V_{ub}| f_+(0)$ from an analysis based on the soft collinear effective theory [53].

RQCD) [56, 57, 58]. The mNRQCD is a generalized version of non-relativistic QCD in the B meson moving frame ($\mathbf{u} = \mathbf{p}_B/M_B \neq \mathbf{0}$). The action is given by

$$\mathscr{L}_{\mathrm{mNRQCD}} = \psi^\dagger \left(iD_t + i(\mathbf{v}\cdot\mathbf{D}) + \frac{\mathbf{D}^2}{2\gamma m} - \frac{(\mathbf{v}\cdot\mathbf{D})^2}{2\gamma m} + \cdots \right) \psi \tag{2.19}$$

where $u_\mu = \gamma(1, \mathbf{v})$ and $\gamma^{-1} = \sqrt{1 - \mathbf{v}^2}$. Setting $\mathbf{v} = \mathbf{0}$ gives the usual NRQCD action. The mNRQCD allows the $B \to \pi l \nu$ calculation at lower q^2 with smaller $\mathbf{p}_\pi$; for $\mathbf{v} \approx 0.75$, $q^2 = 0$ can be achieved with $\mathbf{p}_\pi = 1$GeV, where the size of lattice discretization effects from pion momentum is modest.

Previously lattice simulation with the mNRQCD for a large $\mathbf{v}$ suffered from large statistical errors [56], but Ref. [59] showed that it can be reduced by using a special smearing function so that the simulation with $\mathbf{v} \approx 0.7$–0.8 is feasible. Ref. [60] made a quenched study to test the mNRQCD. They calculated the decay constant of a heavy-heavy meson f_{HH} at non-zero momentum $\mathbf{p}$ using both the NRQCD ($\mathbf{v} = \mathbf{0}$) action and mNRQCD ($\mathbf{v} \neq \mathbf{0}$) action. As shown in Fig. 9, f_{HH} with the NRQCD depends on $\mathbf{p}$, indicating that the $\mathbf{p}$-dependent discretization effect is not under control. On the other hand, the result with the mNRQCD is constant in $\mathbf{p}$ as it should be, and the statistical errors are reasonably small. The result looks encouraging, so it may be worth studying the $B \to \pi l \nu$ from factors at lower q^2 using the mNRQCD formalism.

2.2.2 $B \to D l \nu$ and $B \to D^* l \nu$ decays

The form factors of $B \to D l \nu$ and $B \to D^* l \nu$ decays can be calculated more accurately than those of heavy-to-light decays, due to the approximate symmetry between the initial and final states. The branching fraction of the $B \to D l \nu$ decay is given by

$$\mathrm{Br}(B \to D l \nu) = |V_{cb}|^2 \int dw\, F(w) \times (\text{known factor}) \tag{2.20}$$

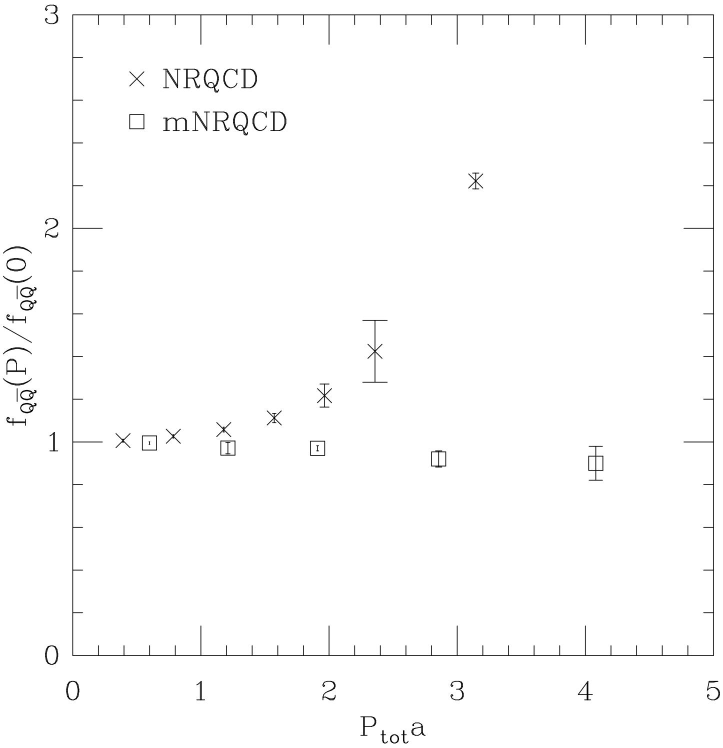

Figure 9: Momentum-dependence of the ratio of heavy-heavy meson decay constants $f_{HH}(\mathbf{p})/f_{HH}(\mathbf{0})$ from NRQCD (crosses) and mNRQCD (squares) [60, 61].

where $w = v_B \cdot v_D$ with $v_B = p_B/m_B$ and $v_D = p_D/m_D$. The form factor of the $B \to Dl\nu$ decay at zero recoil limit, $F_{B\to D}(w=1)$, can be precisely determined by considering the double ratio [62, 63];

$$\frac{C^{DV_0B}(t)C^{BV_0D}(t)}{C^{DV_0D}(t)C^{BV_0B}(t)} \stackrel{t\to\infty}{\longrightarrow} \frac{\langle D|V_0|B\rangle\langle B|V_0|D\rangle}{\langle D|V_0|D\rangle\langle B|V_0|B\rangle}, \tag{2.21}$$

where $C^{DV_0B}(t)$ and $\langle D|V_0|B\rangle$ are respectively the $B \to D$ three-point function and amplitude.

The Fermilab/MILC collaboration has a preliminary $n_f = 2+1$ unquenched result for the $B \to Dl\nu$ form factor using the MILC configurations [21]. The light quark mass dependence of the form factor is mild and a linear chiral extrapolation was made, as shown in Fig. 10. Their result is $F^{B\to D}(1) = 1.074(18)(16)$, which is consistent with the quenched result [62]. Combined with an average of experimental results for $|V_{cb}|F(1)$ [64], one obtains

$$|V_{cb}|_{\mathrm{Lat05}} = 3.91(09)(34)\times 10^{-2}, \tag{2.22}$$

where the error from the lattice calculation (first error) is much smaller than the experimental one (second).

For a more precise determination of $|V_{cb}|$, the $B \to D^*l\nu$ decay should be used because the experimental uncertainty is smaller. As for the lattice calculation, the chiral extrapolation is a key because a singularity should appear for the form factors around the physical pion mass point

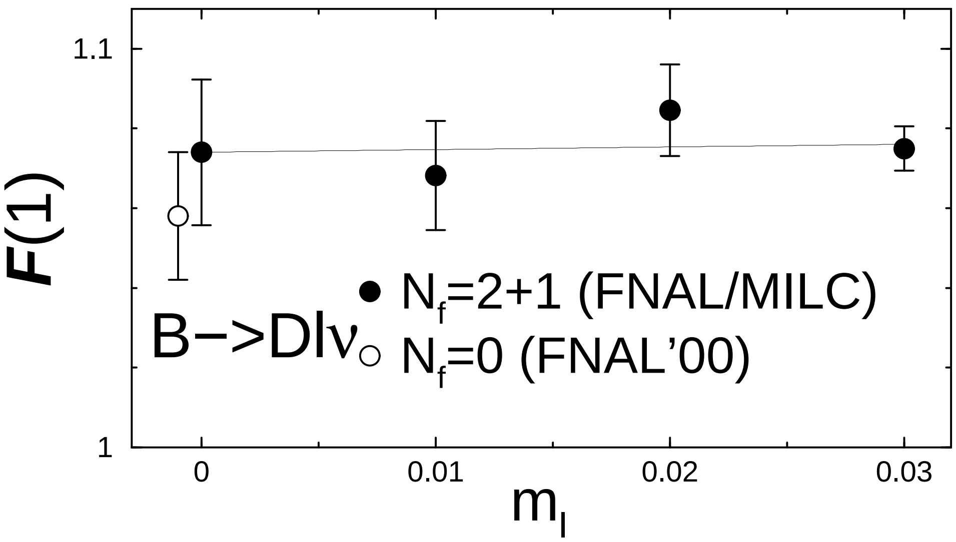

Figure 10: Light quark mass dependence of the unquenched $B \to Dl\nu$ form factor at zero recoil [21]. The quenched result [62] is also shown around at $m_l = 0$

($m_{PS} = m_{D^*} - m_D \simeq m_\pi$). Ref. [65] calculated the $B \to D^* l\nu$ form factors in the staggered chiral perturbation theory (SχPT), and found that the singularity (seen in the continuum χPT) disappears due to lattice discretization effects. The unquenched calculation of the $B \to D^* l\nu$ form factors is underway [65].

2.3 *K* meson decays

The semileptonic $K \to \pi l\nu$ decay is traditionally used to determine $|V_{us}|$. The form factor of $K \to \pi l\nu$ at $q^2 = 0$ can be precisely calculated using the double ratio method similar to the $B \to Dl\nu$ case. The first precise lattice calculation of the $K \to \pi l\nu$ form factor has been done in the quenched approximation using improved Wilson quarks [66]. The quenched result is

$$f_+^{K\to\pi}(0) = 0.960(5)(7) \quad (n_f = 0), \tag{2.23}$$

which agrees with an earlier estimate using a quark model [67]. The uncertainty is smaller than 1% due to the approximate symmetry between initial and final states, as for the $B \to Dl\nu$ form factor. Since then, three unquenched calculations have been started. One is the $n_f = 2$ calculation using improved Wilson quarks by the JLQCD collaboration [68], obtaining the preliminary value $f_+^{K\to\pi}(0) = 0.952(6)$. Another $n_f = 2$ calculation is underway using domain-wall quarks by the RBC collaboration [69]; their preliminary result is $f_+^{K\to\pi}(0) = 0.955(12)$. A preliminary $n_f = 2+1$ calculation on the MILC configurations is done using a combination of improved Wilson and staggered quarks by the Fermilab collaboration [15], getting $f_+^{K\to\pi}(0) = 0.962(6)(9)$. These calculations rely on the chiral perturbation theory to guide the chiral extrapolation $m_l \to m_{ud}$ using data at $m_l > m_s/2$. It may be interesting to perform a direct simulation at smaller m_l using staggered fermions or chirally-improved fermions. For technical details of the calculations, see Ref. [16].

Taking a simple average of three preliminary unquenched results,

$$f_+^{K\to\pi}(0) = 0.956(12) \quad (n_f \geq 2). \tag{2.24}$$

is obtained. Combining this with the experimental result for $|V_{us}|f_+^{K\to\pi}(0)$ [70] gives

$$|V_{us}|_{\rm semi-lep} = 0.2250\ (24)(12), \tag{2.25}$$

where the first error is from lattice calculations and the second from experiment.

$|V_{us}|$ may be determined from the leptonic decay $K \to l\nu$ [71]. The leptonic decay constant f_K has been precisely calculated in $n_f = 2+1$ lattice QCD using improved staggered quarks by the MILC collaboration [22]. Their updated result is [72]

$$f_K/f_\pi = 1.200(4)(^{+17}_{-05}). \tag{2.26}$$

Using this and the experimental result for $\mathrm{Br}(K \to l\nu)/\mathrm{Br}(\pi \to l\nu)$, one obtains

$$|V_{us}|_{\rm lep} = 0.2238(^{+12}_{-32}), \tag{2.27}$$

which is consistent with one from the semileptonic decay Eq. (2.25).

For the Lattice 2005 value, I take an weighted average of Eqs. (2.25) and (2.27), getting

$$|V_{us}|_{\rm Lat05} = \boxed{0.2244(14)}\ . \tag{2.28}$$

There is a 2σ disagreement between Eq. (2.28) and the PDG average $|V_{us}| = 0.2200(26)$ [19]. The reason is as follows. The PDG used the $K \to \pi l\nu$ decay for $|V_{us}|$; the form factor is take from Ref. [67] which is consistent with Eq. (2.24), but the $K \to \pi l\nu$ branching fraction is from earlier experimental measurements which disagrees by $\approx 2\sigma$ with recent ones used for Eq. (2.25).

2.4 Other CKM magnitudes from CKM unitarity

Having the 5 CKM matrix elements determined, one can check a unitarity condition of the CKM matrix using results from lattice QCD alone. From Eqs. (2.14), (2.15) and (2.22), one gets

$$(|V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2)^{1/2} = 1.00(10), \tag{2.29}$$

which is consistent with CKM unitarity.

Conversely one may use CKM unitarity to determine other CKM matrix elements as follows;

$$\begin{aligned} |V_{ud}|_{\rm Lat05} &= (1 - |V_{us}|^2 - |V_{ub}|^2)^{1/2} \\ &= 0.9745(3), & (2.30) \\ |V_{tb}|_{\rm Lat05} &= (1 - |V_{ub}|^2 - |V_{cb}|^2)^{1/2} \\ &= 0.9992(1), & (2.31) \\ |V_{ts}|_{\rm Lat05} &= |V_{us}^* V_{ub} + V_{cs}^* V_{cb}|\ /\ |V_{tb}| \ \simeq\ |V_{cs}^* V_{cb}|\ /\ |V_{tb}| \\ &= 3.79(53)\times 10^{-2}. & (2.32) \end{aligned}$$

One can also determine some of Wolfenstein parameters from Eqs. (2.28), (2.22) and (2.18). One gets

$$\lambda_{\rm Lat05} = |V_{us}| = 0.2244(14), \quad (2.33)$$
$$A_{\rm Lat05} = |V_{cb}|/\lambda^2 = 0.78(7), \quad (2.34)$$
$$(\rho^2+\eta^2)^{1/2}_{\rm Lat05} = |V_{ub}|/(A\lambda^3) = 0.40(8). \quad (2.35)$$

The determination of remaining CKM parameters, (ρ,η) and $|V_{td}|$, will be discussed in next section.

3. CKM phase from lattice QCD

In this section, I extract CKM phase parameters (ρ,η) and the magnitude $|V_{td}|$ using recent lattice results. One constraint on (ρ,η) has already been obtained in previous section, Eq. (2.35). Two more constraints may be obtained from the mixing of neutral $B_{(s)}$ and K mesons.

The neutral $B_{(s)}$ mixing is characterized by the mass difference of $B^0_{(s)}$ and $\bar{B}^0_{(s)}$ mesons, which is given by

$$\Delta M_{B_q} = (\text{known factor}) \times f^2_{B_q} B_{B_q} |V^*_{tb} V_{tq}|^2 \quad (q=d,s) \quad (3.1)$$

The nonperturbative QCD effects are contained in $f^2_{B_q}B_{B_q}$, where f_{B_q} is the B_q meson decay constant defined in an analogous way to Eq. (2.3) and B_{B_q} is the B_q meson bag parameter defined through

$$\langle \bar{B}^0|(\bar{b}q)_{V-A}(\bar{b}q)_{V-A}|B^0\rangle \equiv \frac{8}{3} m_{B_q} B_{B_q} f^2_{B_q} \quad (q=d,s) \quad (3.2)$$

By combining the lattice calculation of $f^2_{B_d}B_{B_d}$ with the precisely measured value of ΔM_{B_d}, one can extract $|V^*_{tb}V_{td}| \simeq |V_{td}| = A\lambda^3\sqrt{(1-\rho)^2+\eta^2}$, which gives a constraint on (ρ,η). Since some uncertainties for lattice results are canceled in the ratio $\frac{f^2_{B_d}B_{B_d}}{f^2_{B_s}B_{B_s}}$, it is desirable to consider the ratio of B_d and B_s meson mass differences,

$$\frac{\Delta M_{B_d}}{\Delta M_{B_s}} = \frac{M_{B_d}}{M_{B_s}} \frac{f^2_{B_d}B_{B_d}}{f^2_{B_s}B_{B_s}} \frac{|V^*_{tb}V_{td}|^2}{|V^*_{tb}V_{ts}|^2} \propto \frac{|V_{td}|^2}{|V_{ts}|^2} = \lambda^2[(1-\rho)^2+\eta^2]. \quad (3.3)$$

Up to now, however, only the lower limit is known for ΔM_{B_s}. The CDF and D0 experiments are expected to measure ΔM_{B_s}; once it is available, it will provide a better constraint on (ρ,η).

The neutral K mixing is characterized by the CP-violating parameter ε_K, given by

$$|\varepsilon_K| = B_K\,\eta[(1-\rho)c_1 + c_2] \quad (3.4)$$

where c_1 and c_2 are numerical constants, and B_K is defined through

$$\langle \bar{K}^0|(\bar{s}d)_{V-A}(\bar{s}d)_{V-A}|K^0\rangle = \frac{8}{3} m_K B_K f^2_K, \quad (3.5)$$

with f_K being the K meson decay constant. Combining the lattice result for B_K and the experimental result for $|\varepsilon_K|$ gives another constraint on (ρ,η).

As for other constrains on (ρ,η), I note that recent experiments by B factories enable to measure all 3 angles of the CKM unitarity triangle $(\alpha,\beta,\gamma) = (\arg\left(\frac{V_{tb}^* V_{td}}{V_{ub}^* V_{ud}}\right), \arg\left(\frac{V_{cb}^* V_{cd}}{V_{tb}^* V_{td}}\right), \arg\left(\frac{V_{ub}^* V_{ud}}{V_{cb}^* V_{cd}}\right))$. In particular, the accuracy of β is impressive. See, for example, Refs. [73, 74, 12] for recent experimental measurements of the CKM angles.

The constraints, Eqs. (2.35), (3.1) and (3.4), as well as other constrains, over-determine (ρ,η). If a good precision can be achieved for each sector, one can test the Standard Model by seeing whether or not there is inconsistency between them. Below I review recent lattice results for $f_{B_{d(s)}}$, $B_{B_{d(s)}}$ and B_K, and then extract (ρ,η) using them.

3.1 B meson mixing ($f_B^2 B_B$)

3.2 f_B

I first remind that f_B is similar to f_D and lattice results for the latter agree with the experimental result by CLEO-c within $\approx$10% uncertainties, as seen in previous section. This may indicate reliability of lattice results for f_B.

This year the HPQCD collaboration finalized their $n_f = 2+1$ calculation of f_B using the improved staggered quarks and the NRQCD heavy quark on the MILC configurations [75, 76, 77, 78]. They performed simulations at the light quark masses ranging $m_s/8 \le m_l \le m_s/2$. The chiral extrapolations are made using various fit forms including ones with the staggered χPT, the continuum χPT and a simple linear ansatz. They reported that the various fits agree with each other within 3% after the extrapolation, suggesting that the results at the physical light quark mass ($m_l = m_{ud}$) are insensitive to details of the chiral fit.

The $n_f = 2+1$ result by HPQCD collaboration is shown in Fig. 11 together with the $n_f = 2$ result by JLQCD collaboration [79]. Although the JLQCD result is consistent with the HPQCD result at $m_l \simeq m_s/2$, a linear chiral extrapolation using only JLQCD data (ranging $m_l \ge m_s/2$) clearly deviates from the HPQCD result at $m_l = m_{ud}$. I believe that this is an evidence that data at $m_l \le m_s/2$ is required for high-precision lattice calculations. The result by the HPQCD collaboration is [75, 78]

$$f_{B_d} = 216(09)(19)(07) \text{ MeV}, \tag{3.6}$$

$$f_{B_s} = 260(07)(26)(09) \text{ MeV}, \tag{3.7}$$

where the first errors are statistical one and uncertainties from the chiral extrapolation, the second are ones from the 1-loop perturbative matching for the current renormalization, and the third are other uncertainties. The 2-loop or nonperturbative matchings will be required for an accuracy better than 5%. The uncertainties from the matching (and some others) are canceled in the ratio of B_d and B_s decay constants, giving

$$f_{B_s}/f_{B_d} = 1.20(3)(1). \tag{3.8}$$

This is a 3% determination, thanks to the small error from the chiral extrapolation. Once Δm_{B_s} is measured, this will significantly reduce uncertainty for $|V_{ts}|/|V_{td}|$.

Comparing the HPQCD results with the latest averages of unquenched results ($n_f \ge 2$) in Ref. [80], one sees a reasonable agreement for individual f_{B_d} and f_{B_s}, and a good agreement for

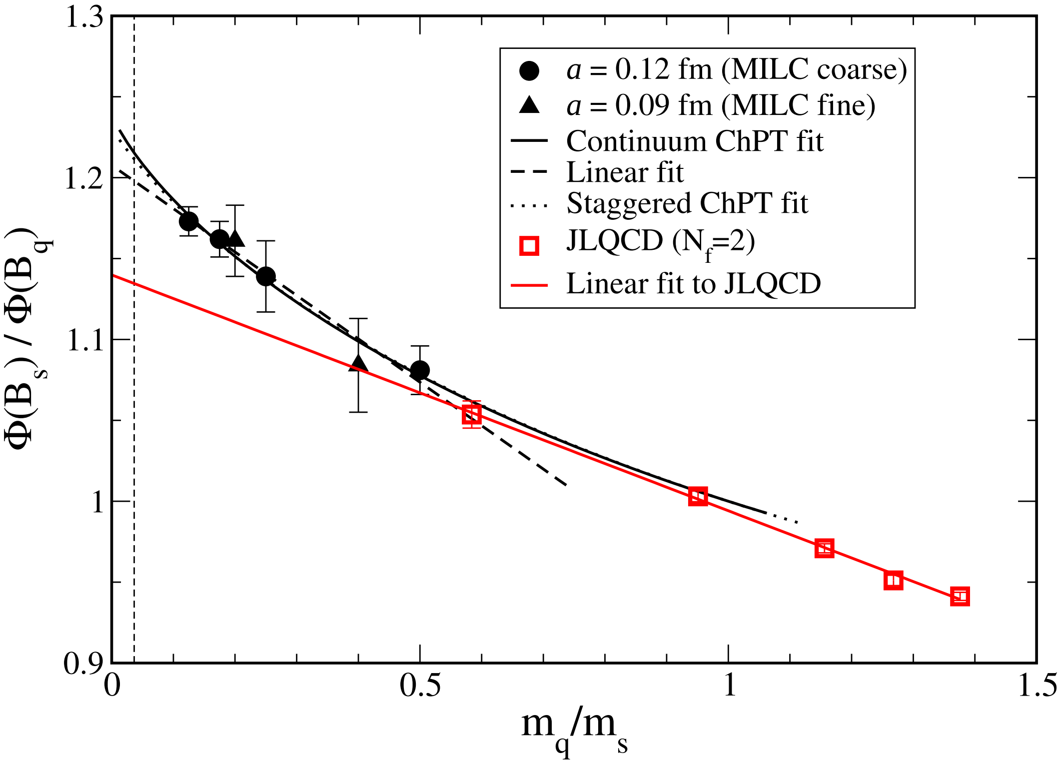

Figure 11: The light quark mass dependence of unquenched $f_{B_s}\sqrt{m_{B_s}}/(f_{B_d}\sqrt{m_{B_d}})$ by the HPQCD collaboration ($n_f = 2+1$, black symbols) [75] and by the JLQCD collaboration ($n_f = 2$, red symbols) [79]. Black (red) lines are the chiral extrapolations using the HPQCD (JLQCD) data. The dashed vertical line indicates the physical light quark mass ($m_l = m_{ud}$).

the ratio f_{B_s}/f_{B_d}, as shown in Fig. 12. The good agreement for the ratio is probably due to the cancellation of some systematic uncertainties, as mentioned above. I also note that the averaged value in Ref. [80] is estimated by including the effect of chiral logarithm from χPT to guide the chiral behavior for $m_l \leq m_s/2$.

3.2.1 B_B

Turning to B_B, there is no new or updated result in unquenched QCD this year.[3] The $n_f = 2$ calculation by the JLQCD collaboration using an improved Wilson light quark and the NRQCD heavy quark [79] is still only result from unquenched QCD. Their result is

$$B_{B_d}(m_b) = 0.836(27)(^{+56}_{-62}), \tag{3.9}$$

$$B_{B_s}/B_{B_d} = 1.017(16)(^{+56}_{-17}). \tag{3.10}$$

[3] While this paper is being completed, a new unquenched ($n_f = 2$) calculation using domain-wall light quarks and an improved static heavy quark has been reported [81]. The results for B_{B_s}/B_{B_d} and $f_{B_s}/f_{B_d}\sqrt{B_{B_s}/B_{B_d}}$ are consistent with but larger than Eqs (3.10) and (3.12).

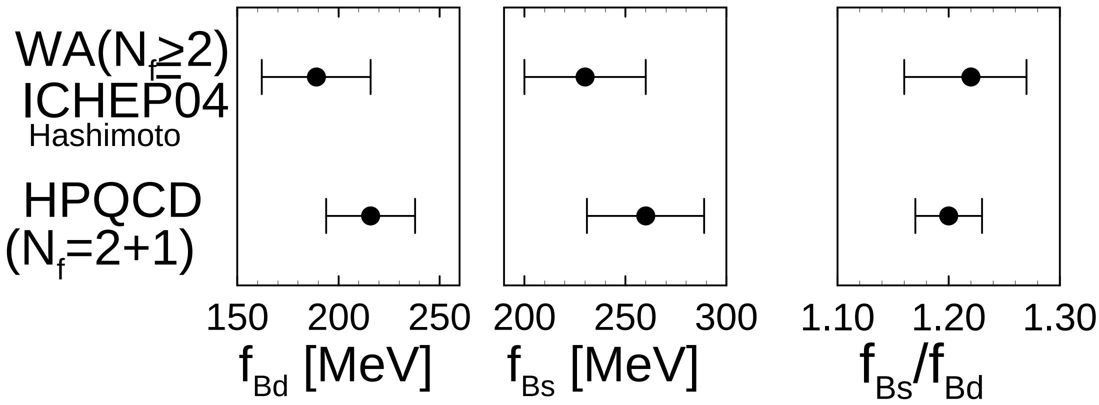

Figure 12: Comparison of the B meson decay constants f_B in $n_f = 2+1$ QCD by the HPQCD collaboration [75] and the unquenched world average in Ref. [80].

Combining this and the HPQCD result for f_B gives

$$f_{B_d}\sqrt{\hat{B}_{B_d}} = 244(26)\ \text{MeV}\ , \tag{3.11}$$

$$f_{B_s}/f_{B_d}\sqrt{B_{B_s}/B_{B_d}} = 1.210(^{+47}_{-35}). \tag{3.12}$$

where $\hat{B}_B$ is the renormalization group invariant bag parameter. These values should be compared with the previous average, *e.g.,* Ref. [80] quoted $f_{B_d}\sqrt{\hat{B}_{B_d}} = 214(38)$ and $f_{B_s}/f_{B_d}\sqrt{B_{B_s}/B_{B_d}} = 1.23(6)$. Equation (3.11) together with the experimental value of Δm_{B_d} leads to

$$|V_{td}|_{\text{Lat05}} = 7.40(79)\times 10^{-3}, \tag{3.13}$$

which is consistent but smaller than the PDG value [19], $|V_{td}|_{\text{PDG}} = 8.3(1.6)\times 10^{-3}$. Equation (3.12) and the forthcoming measurement of Δm_{B_s} will give a 3–4% determination of $|V_{ts}|/|V_{td}|$.

The bag parameters are also studied in quenched ($n_f = 0$) QCD using the overlap light quark action [82], and in perturbation theory using the twisted mass light quark action [83, 84]. Using these actions has an advantage that operator mixings do not occur (or can be removed), which may lead a more precise calculation of the bag parameters in the future. For previous quenched results, see, *e.g.,* Ref. [7].

3.3 K meson mixing (B_K)

Three new studies of B_K in $n_f = 2+1$ unquenched QCD are reported this year [85, 86, 87]. In particular, the HPQCD collaboration presented a preliminary value using improved staggered quarks on the MILC configurations [85],

$$B_K^{\bar{MS}}(2\text{GeV}) = 0.630(18)(130)(34), \tag{3.14}$$

where the first error is statistical, the second is from the 1-loop matching which is again the largest error, and the third is other uncertainties. Ref. [86] also used an improved staggered fermion on the

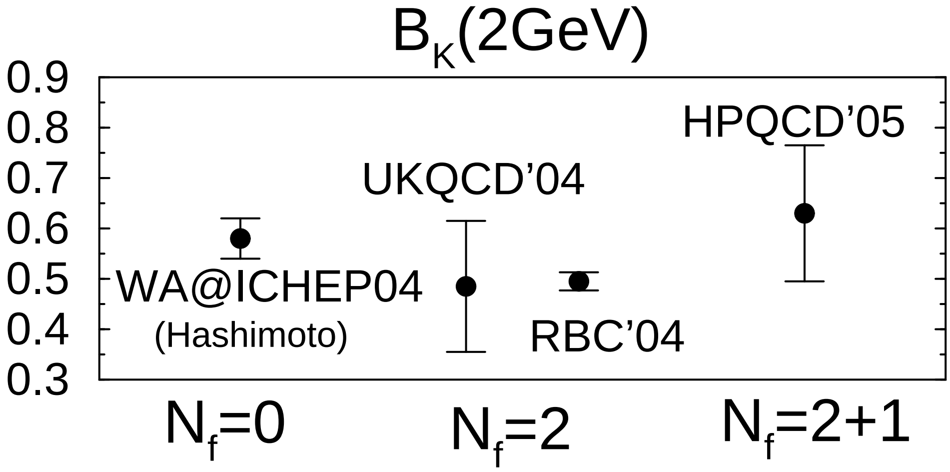

Figure 13: Comparison of recent unquenched results for B_K [85, 88, 89] and the average of quenched results [80]. The $n_f = 2$ result by the UKQCD collaboration is obtained with an improved Wilson fermion action [88], and one by the RBC collaboration is obtained with a domain wall fermion action [89]. The error for the result by the RBC collaboration is statistical only.

same configurations. On the other hand, Ref. [87] used the domain wall fermion for both valence and dynamical quarks. At present the latter two groups do not quote the physical value of B_K. For details of recent B_K calculations, see Ref. [16].

A comparison of recent unquenched results [85, 88, 89] and the average of quenched results [80] is shown in Fig. 3.3. The unquenched results are consistent with the quenched average, but uncertainties are much larger. Given this, I do not take the average for the unquenched B_K, and simply use Eq. (3.14) as a representative of unquenched results in the following unitarity triangle analysis.

3.4 Unitarity triangle analysis

Let us now analyze the unitarity triangle using recent lattice results to extract $(\bar{\rho}, \bar{\eta})$. As mentioned before, I use unquenched lattice results only as the theory input. Here I adopt Eq. (2.35) for $(\rho^2 + \eta^2)^{1/2}$ from the $B \to \pi l \nu$ form factor, Eq. (3.11) for $f_{B_d}\sqrt{\hat{B}_{B_d}}$, and Eq. (3.14) for B_K.

As the experimental inputs, I use ΔM_{B_d} and $|\varepsilon_K|$. The $B \to \pi l \nu$ branching fraction $\mathrm{Br}(q^2 \geq 16\ \mathrm{GeV}^2)$ is also used to obtain Eq. (2.35). Using the lower limit of $\Delta M_{B_s}/\Delta M_{B_d}$ together with Eq. (3.12) for $f_{B_s}/f_{B_d}\sqrt{B_{B_s}/B_{B_d}}$ does not affect the result of this analysis for the reason given below. I also use the result for $\sin(2\beta)$ (=0.726(37) [64]) from $B \to (c\bar{c})K^{(*)}$ decays to see its impact on the $(\bar{\rho}, \bar{\eta})$ determination.

The unitarity triangle using lattice results together with the $\sin(2\beta)$ constraint is shown in Fig. 14. The shaded regions indicate 1σ error bands. A difference between this and previous ones (*e.g.,* one in Ref. [19]) is that the position of the ΔM_{B_d} bound moved to right with a smaller uncertainty. Consequently, the bound from the lower limit of $\Delta M_{B_s}/\Delta M_{B_d}$ does not change the result for $(\bar{\rho}, \bar{\eta})$.

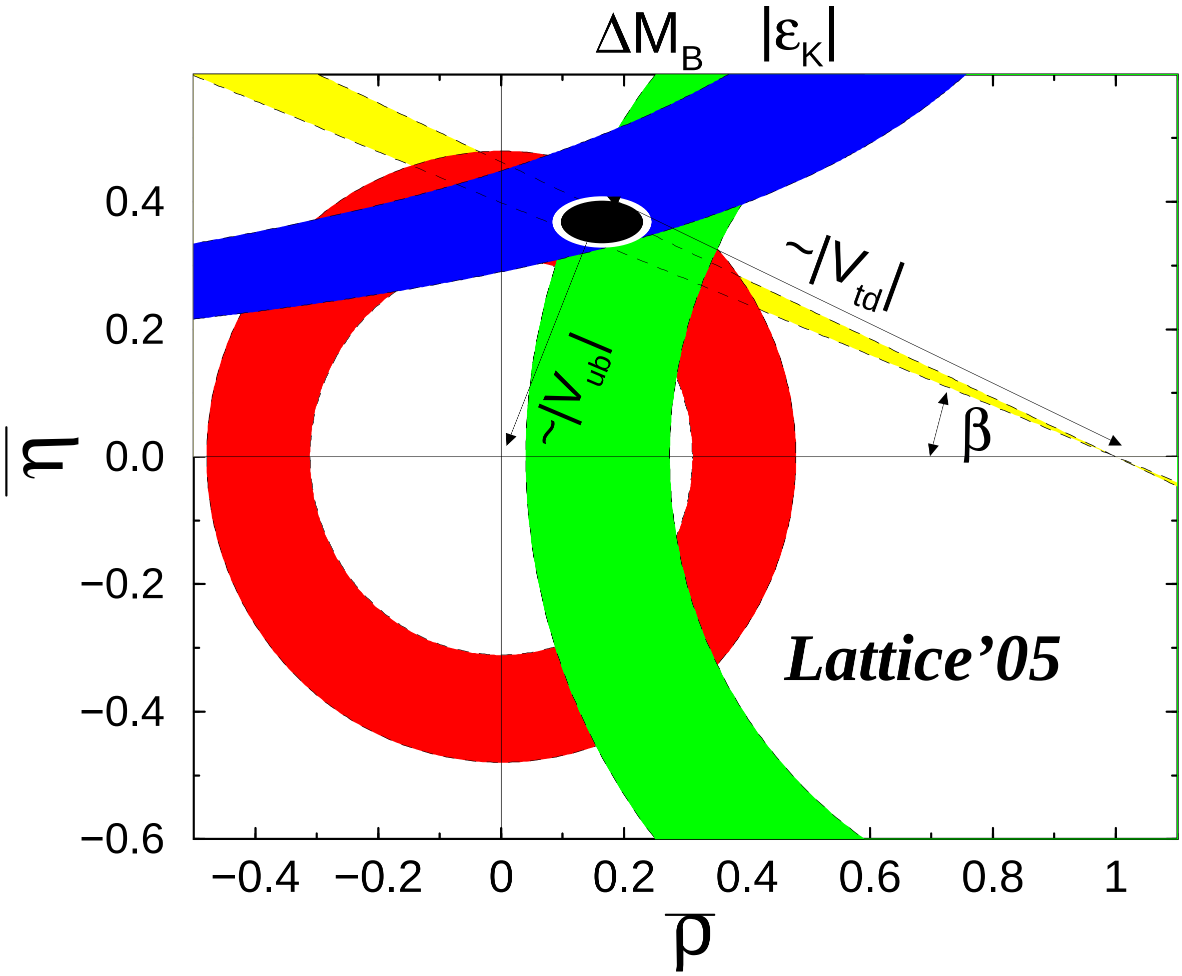

Figure 14: Unitarity triangle using recent unquenched lattice results for constraints from $|V_{ub}|$ (red), ΔM_B (green) and ε_K (blue). The $\sin(2\beta)$ constraint (yellow) is also shown. The shaded regions indicate 1σ error bands.

From the overlapped region in $(\bar{\rho},\bar{\eta})$ plane, I read

$$\bar{\rho} = 0.20(12), \tag{3.15}$$
$$\bar{\eta} = 0.37(07) \tag{3.16}$$

without the $\sin(2\beta)$ constraint. Including the $\sin(2\beta)$ constraint significantly improves the precision, giving

$$\bar{\rho}_{\rm Lat05} = 0.16(7), \tag{3.17}$$
$$\bar{\eta}_{\rm Lat05} = 0.37(4), \tag{3.18}$$

which are consistent with the PDG values [19] $\rho = 0.20(9)$ and $\eta = 0.33(5)$, and the accuracy is comparable, as shown in Fig. 15.

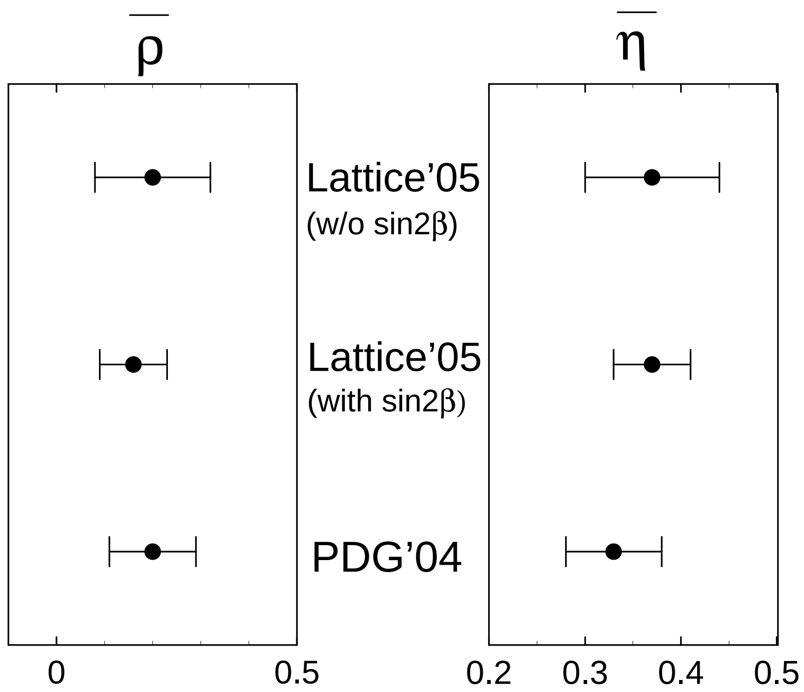

Figure 15: Comparison of $(\bar{\rho}, \bar{\eta})$ using recent unquenched lattice results (without and with the $\sin(2\beta)$ constraint) and one quoted in Ref. [19].

4. Conclusion

This paper presents a full determination of the CKM matrix using recent lattice results for gold-plated quantities. To extract the CKM matrix elements in a uniform fashion, results from unquenched lattice QCD are exclusively used as the theory input for nonperturbative QCD effects. The results for the CKM matrix elements are Eqs. (2.14), (2.15), (2.18), (2.22), (2.28), (2.30), (2.31), (2.32) and (3.13). The results for the Wolfenstein parameters are Eqs. (2.33), (2.34), (3.17) and (3.18). These are summarized in one place, Eqs. (1.2)-(1.6).

At present, many unquenched results are obtained with improved staggered fermion actions. On the other hand, fewer unquenched results are obtained with other fermion formalisms (such as the Wilson-type fermion, domain wall fermion and overlap fermion), especially for the $n_f = 2+1$ case. Consequently, the results for the CKM matrix elements presented here are often estimated from only one or two unquenched calculations. I expect that more unquenched results using other lattice fermions will come out in the near future, leaving future reviewers to make a more serious average of the CKM matrix.

The unquenched results for D physics (such as the $D \to \pi l \nu$ form factor $f_+^{D\to\pi}$ and the D meson decay constant f_D) are in agreement with recent experimental results. This may increase confidence in lattice results for similar quantities for B physics (such as the $B \to \pi l \nu$ form factor and f_B).

The typical accuracy of most of gold-plated quantities is $O(10\%)$, dominated by uncertainties from the lattice discretization effects, perturbative matching, and the chiral extrapolation. The

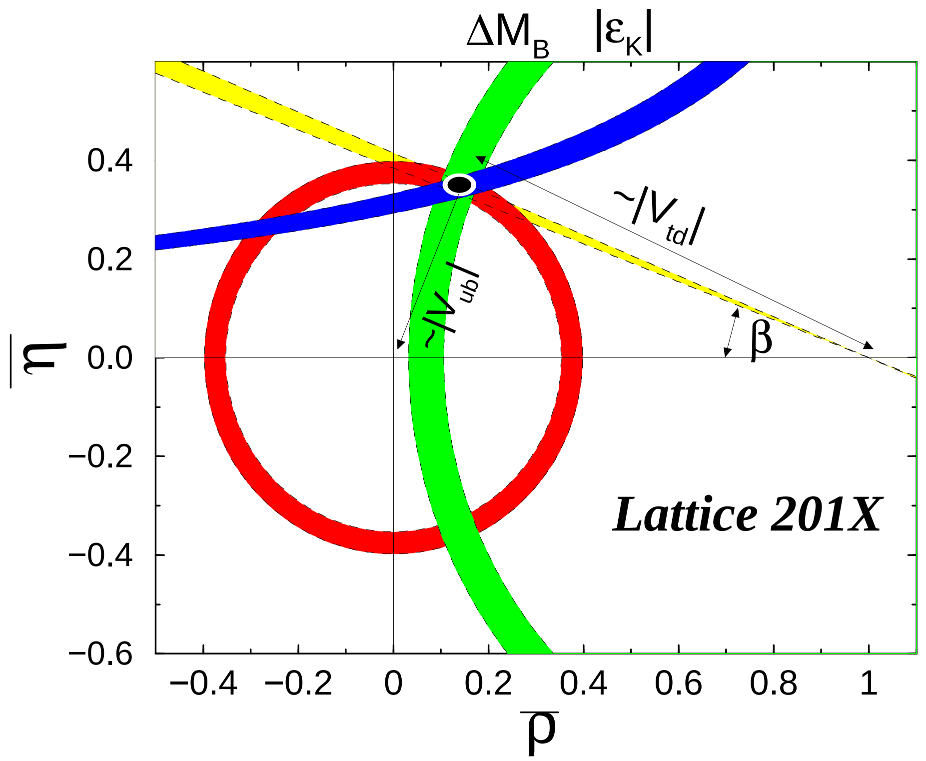

Figure 16: Expected unitarity triangle with 5% accuracies for the $B \to \pi l \nu$ form factor and B_K and a 3% accuracy for $f_{B_s}/f_{B_d}\sqrt{B_{B_s}/B_{B_d}}$. The ΔM_{B_s} measurement is also assumed here.

last uncertainty is especially for lattice fermions other than the staggered fermion. To achieve an accuracy better than 5%, simulations at smaller lattice spacings and smaller quark masses and higher-order matchings will be required.

A better accuracy (3% or less) is obtained for the $B \to Dl\nu$ and $K \to \pi l \nu$ form factors due to the approximate symmetry between initial and final states, and for the decay constant ratio f_{B_s}/f_B due to the cancellation of systematic errors. The latter will lead to a more precise constraint on (ρ, η) once ΔM_{B_s} is measured by experiment.

Assuming the ΔM_{B_s} measurement and 5% (or better) accuracies for lattice results, the unitarity triangle will be something like Fig. 16. I hope that this will be realistic in next 5 years so that we can more precisely test the standard model using lattice QCD and be ready for new physics.

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