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On the Casimir Effect for N-dimensional Spheres

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The purpose of this work is to show explicitly, that the use of an exponential cut-off regularization scheme in the calculation of Casimir energy densities in S^N may lead to conflicting results when compared to those found with the aid of zeta function methods. The main issue is that the latter scheme always delivers finite results, while the former produces some extra divergent terms which are not of easy interpretation

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1. Introduction

The great success of quantum field theory in the description of the microscopic world cosmos is widely recognized nowadays, but it surely did not happen without great effort. A remarkable breakthrough took place when we learned how to extract physical results from the infinities that appeared in the calculations by a complex procedure known generally as renormalization[1][2][3][4][5][6]. Without going into further details, we might only add that a renormalization program usually requires a regularization prescription, and even that preliminary step has its subtleties. It is illustrative to consider the quantum electrodynamics case, where it took decades to go from its birth[7] to the all-order renormalization proof[6][8].

Being renormalization such a difficult problem, it is certainly convenient studying it on its simplest manifestation. So, in order to explore how the regularization and renormalization processes work in curved manifolds, we compute the Casimir energy for a massless scalar field living on the S^N (which is probably the simplest example of curved manifold). The computation is done by using two different regularization methods, and the nature of the results will force us to pay closer attention into one of them (exponential cut-off) if $N \geq 4$. It is important to stress at this point that these results are known in the literature [14][13][15], we are just looking at them from another perspective.

Last but not least, it must be said that the study of field theory on the S^N has some history. A very thorough discussion on the subject is given in [14], and Casimir effect calculations were firstly done (as long as we are aware of) in [15], followed by [16][12][13][17], among others. There were also a good deal of activity on the S^N Casimir problem within the framework of multi-dimensional Kaluza-Klein theories, see [18][19][20][21].

2. The Casimir energy

In order to get some insight on the problem, let us begin by solving the equation of motion for a massive scalar field on the S^N. The Klein-Gordon equation conformally coupled for a curved manifold is[1] [11]

$$\nabla_\nu \nabla^\nu \phi + m^2 \phi + \frac{N-1}{4N} R\phi = 0 \tag{2.1}$$

where ∇_ν is the covariant derivative and R is the curvature scalar.

The metric element on the S^N is

$$ds^2 = dt^2 - a^2 d\Omega^2 \qquad \left(a = \frac{1}{R}\right) \tag{2.2}$$

where a is the radius and $d\Omega$ is the solid angle on the hypersphere and has a rather cubersome expression [22] for generic dimension. Using (2.2) to evaluate the covariant derivatives and the result[2]

[1] the units are such that $\hbar = c = 1$.

[2] We are using the same conventions of Birrell and Davies [11].

$$R_{(S^N \otimes T)} = \frac{N(N-1)}{a^2} \tag{2.3}$$

we arrive at

$$\left(\partial_\nu \partial^\nu + m^2 - \frac{1}{a^2} \nabla^2_{ang} + \frac{1}{a^2} \frac{(N-1)^2}{4} \right) \phi = 0 \tag{2.4}$$

where ∇^2_{ang} is the angular part of the laplacian in N dimensions. Separating variables, we get

$$\left(\partial_t^2 + m^2 + \frac{\Lambda^2}{a^2} + \frac{(N-1)^2}{4a^2} \right) \phi_t(t) = 0 \tag{2.5}$$

$$\nabla^2_{ang} \, \phi_{ang}(\theta_1, \theta_2, ...\theta_{N-1}) = -\Lambda^2 \, \phi_{ang}(\theta_1, \theta_2, ...\theta_{N-1}) \tag{2.6}$$

The second equation, although a bit complicated, has a tabulated solution [22], and below we give only the eigenvalues and its degeneracies

$$\Lambda_l^2 = l(l+N-1) \qquad D_N(\Lambda_l) = (2l+N-1)\frac{\Gamma(l+N-1)}{\Gamma(N)\Gamma(l+1)} \qquad (l \in \mathbb{Z}) \tag{2.7}$$

By a fortunate conspiracy of the curvature scalar and the eigenvalues, the expression (2.5) may be recast into

$$\left(a^2 \partial_t^2 + a^2 m^2 + \left[l + \frac{N-1}{2} \right]^2 \right) \phi(t) = 0 \tag{2.8}$$

which has a well suited form for our purposes. We shall now proceed to the evaluation of the Casimir energy, whose standard definition goes like [10]

$$E_c = \int \langle 0 | T_{00} | 0 \rangle = \frac{1}{2} \sum_l \omega_l \tag{2.9}$$

where of course $T_{\mu\nu}$ is the energy-momentum tensor. Actually, the second equality is not part of the definition and does not always hold, but in this context it is valid. For the sake of simplicity, from now on we will restrict ourselves to the massless scalar field case, for which the dispersion relation reads

$$w_l = \frac{1}{a} \left(l + \frac{N-1}{2} \right) \tag{2.10}$$

and the Casimir energy then becomes

$$E_c = \frac{1}{2} \sum_l D_N(\Lambda_l) \omega_l = \frac{1}{2a} \sum_l (2l+N-1) \frac{\Gamma(l+N-1)}{\Gamma(N)\Gamma(l+1)} \left(l + \frac{N-1}{2} \right) \tag{2.11}$$

The last expression diverges, being meaningless as it stands. In order to give a proper meaning to this formula, we will have to carry on some procedure of regularization and renormalization, and that is the central issue of this work. We are going to present now two different types of regularization and then make a comparison between the results achieved with them[3]

[3] This was done on very general lines in [10], but only to the 3-dimensional case.

We may start by introducing an exponential cut-off on the expression (2.11)

$$E_c(\delta) = \frac{1}{2}\sum_l D_N(\Lambda_l)\omega_l e^{-\delta\omega_l} = \frac{1}{2a}\sum_{l=0}^{\infty} D_N(\Lambda_l)\left(l+\frac{N-1}{2}\right)e^{-\delta\left(l+\frac{N-1}{2}\right)} \tag{2.12}$$

where δ is our regularizing parameter. The interesting results will arise when $N \geq 4$, and for that reason we may choose $N = 5$ as our initial example

$$E_c^{S^5}(\delta) = \frac{1}{24a}\sum_{l=0}^{\infty}(l+3)(l+2)^3(l+1)e^{\frac{-\delta(l+2)}{a}} \tag{2.13}$$

The above summation may be carried out with the result

$$E_c^{S^5}(\delta) = \frac{1}{24a}\left(-a^5\frac{d^5}{d\delta^5}+a^3\frac{d^3}{d\delta^3}\right)\left[\frac{e^{\frac{\delta}{2a}}}{2\,sinh\left(\frac{\delta}{2a}\right)}\right] \tag{2.14}$$

We are interested in the $\delta \to 0$ limit, so it will be interesting to expand expression (2.14) around $\delta = 0$. After a little bit of algebra we arrive at

$$E_c^{S^5}(\delta) \backsim \frac{5a^5}{\delta^6} - \frac{a^3}{4\delta^4} - \frac{31}{60480a} + O(\delta^2) \tag{2.15}$$

which is the regularized but non-renormalized expression for the Casimir energy.

Let us now halt this calculation and introduce another regularization scheme, namely,

$$E_c(s) = \frac{1}{2}\sum_l D_N(\Lambda_l)\omega_l^{-s} \tag{2.16}$$

where a negative power cuts-off the contribution of the higher modes and regularizes expression (2.16). For $N = 5$ it becomes

$$E_c(s) = \frac{1}{24}\sum_{l=0}^{\infty}(l+3)(l+2)^2(l+1)\left[\frac{1}{a}(l+2)\right]^{-s} = \frac{a^s}{24}\sum_{l=1}^{\infty}\left[l^2(l+1)^{-s+2}+2l(l+1)^{-s+2}\right] \tag{2.17}$$

There is a fairly simple technique to construct the analytic continuation of expressions like the one on the r.h.s. of equation (2.17). It is described in detail on reference [13], and its direct application to this particular case gives

$$E_c(-1) = \frac{1}{24a}\left[24+6\sum_{k=0}^{\infty}\frac{1}{k!\Gamma(4-k)}[\zeta(k-5)+2\zeta(k-4)-3]\right] \tag{2.18}$$

where $\zeta(x)$ is the well known Riemann zeta function. It is easy to see that only a finite number of terms in the summation will contribute (namely, only if k=0,1,2,3,5 or 6). Collecting these terms we arrive at the following result

$$E_c(-1) = -\frac{31}{60480\,a} \tag{2.19}$$

Let us notice that no 'regularization-dependent' terms appear on (2.19). It happens that by doing an analytical continuation one is implicitly disregarding several contributions, so that a renormalization process already took place. No further subtractions are necessary because it turned out that in this case the analytic continuation took care of all the non-physical terms.

Let us now compare the expressions (2.15) and (2.19). It is readily seen that the third term on the r.h.s. of (2.15) is in perfect agreement with the result given by (2.19), but there are two terms that independently blow up when $\delta \to 0$. The first one is a volume[4] term, representing as such an uniform radius-independent energy density contribution. Furthermore, a quick calculation shows that

$$\left[\textit{Hyperarea of } S^5\right] \ X \ \left[\begin{array}{c}\textit{regularized background energy}\\ \textit{density of the five-dimensional}\\ \textit{Minkowski space}\end{array}\right] = \frac{5a^5}{\delta^6} \tag{2.20}$$

which states that this contribution is just a background energy reference and may be subtracted [10]. But there is still missing an explanation for the a^3 term. A possible way to provide such an explanation is by adding a gravitational part to our original action [11]

$$S_T = S_g + S_\phi = \frac{1}{8\pi G}\int (R - 2\Lambda) d^4x \ + \ \int (\partial_\mu\phi\partial^\mu\phi + m^2\phi^2 + \xi R\phi) d^4x \tag{2.21}$$

and then make use of the constants G (Newton's constant) and Λ (cosmological constant) to absorb some divergences. This is done by recognizing the geometrical character of the extra divergent terms that are present in a calculation on a curved background[5]. This approach solves the renormalization problem in this case (S^5), but on the other hand does not explain why we are compelled to invoke a gravitational background field if we are using an exponential cut-off regularization[6].

Besides, that is not the only problem. Let us take a look on the situation for a greater value of N, say, $N = 6$. The manipulations are absolutely identical to the S^5 case and therefore will be skipped. The analogous of expression (2.15) for the S^6 is given by

$$E_c^{S^6}(\delta) \backsim \frac{6a^5}{\delta^7} - \frac{a^4}{2\delta^5} - \frac{3a^2}{320\delta^2} + O(\delta) \tag{2.22}$$

where we have three different radius-dependent terms that diverge in the limit $\delta \to 0$. As a matter of fact, one may show that the number of such terms grows as the dimensionality of the problem increases[7], thus necessarily producing an upper bound for the dimensionality in which an extraction of a physical result by this procedure is possible. In the meantime, the negative power cut-off for the S^6 gives

$$E_c^{S^6} = 0 \tag{2.23}$$

[4]Five-volume actually.

[5]A long and thorough discussion about the renormalization of G and Λ is to be found on chap. 6 of [11]. By this point of view, the subtraction of volume term is merely a renormalization of the cosmological constant.

[6]Here it is important to remark that, even for some 4-dimensional curved space-times, the inclusion of a gravitational action like the one that appears in (2.21) does not suffices to renormalize the theory, see also [11].

[7]It comes directly from the form of the degeneracy of the angular modes. As an alternative, one can also arrive at the same conclusions by making an analysis of the heat kernel expansion [23].

which shows again the power of the analytical continuation to generate divergence-free expressions.

3. Final discussions

It has been shown that two different regularization schemes may deliver different bare results. There is nothing new or surprising about this statement [10], as long as those schemes lead to the same physical expression. Considering the results presented in the preceding section, one can see that this would be exactly the case if we could get rid of some contributions produced by the exponential cut-off procedure.

Unfortunately there seems to be no simple justification for the removal of the undesired terms, but a tentative explanation would be the following. In his seminal paper [9], H. Casimir introduced a cut-off regularization that works very similarly to the exponential cut-off that we are using. This choice of regularization prescription was based on the argument that no real plate is a perfect conductor. The cut-off would then simulate a finite conductivity of the plates, producing thus a very elegant and physical regularization. But all of this reasoning does not apply here, because we are dealing with a compact manifold without boundaries. Perhaps in such cases, cutting-off high frequencies exponentially will not be enough to yield a finite result and techniques based on analytic continuation will be preferable.

We may conclude this paper by saying that whatever explanation we find for this discrepancy, the fact is that calculations of Casimir energies for a massless scalar field on the S^N ($N \geq 4$) provide simple examples for which the cut-off regularization procedure and the zeta function method do not totally agree.

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