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How to Quantize κ-Deformed Field Theories?

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Field theories whose space-time symmetries are governed by the κ-deformed Poincaré algebra exhibit peculiar properties which can be used to study physics at very short scales. Those properties also rise interesting questions about how to quantize such theories. We present some results on the quantization of κ-deformed field theories which can shed some light on those questions.

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The κ-deformed Poincaré algebra is a Hopf algebra, alternatively known as quantum algebra or quantum group, obtained by contracting a deformed anti-de-Sitter algebra [1, 2, 3]. The parameter κ is a positive real number with the dimension of mass, which establishes a fundamental scale for the deformed Poincaré algebra. In the limit $\kappa \to \infty$ the deformation disappears and the usual Poincaré algebra is obtained from its κ-deformed version. The first Casimir invariant of the κ-deformed Poincaré algebra gives rise to the following dispersion relation

$$\mathbf{P}^2 - \left(2\kappa \sinh \frac{P^0}{2\kappa}\right)^2 = -m^2 \,, \tag{1}$$

where P^0 and $\mathbf{P}$ are the energy and momentum generators in the algebra and m^2 is an invariant scalar labeling the considered representation. The usual relativistic dispersion relation is recovered from (1) in the limit $\kappa \to \infty$. The κ-deformation may provide a theoretical framework to describe physics with a new fundamental scale in which violations of usual Poincaré symmetries can be expected. Actually, it has been shown that a single deformation of usual relativistic dispersion relation can explain experimental paradoxes presently facing the astrophysics community [4] (see [5] for a review on the subject and its relation with κ-deformation). It is plausible to expect violations of usual Poincaré symmetry at very large scales of energy or very small scales of length, which faces us with the challenge of constructing quantum field theories whose space-time symmetries are governed by the κ-deformed Poincaré algebra. A κ-deformed quantum field theory has been proposed in the framework of non-commutative space-time by Kosinski, Lukierski and Maslanka [6]. They considered a self-interacting theory and obtained as a consequence of the κ-deformation that four-momentum is not conserved at the vertexes. Here we want to consider κ-deformed quantum field theories in usual space-time with no interactions but eventually under external influences simulated by boundary conditions. The construction of complete theories of quantum κ-deformed fields is far from trivial and we want to present here some results in this direction, discussing the peculiarities of such theories that can be used not only to study new physics but also rise interesting questions about their quantization.

In a κ-deformed field theory the equations of motion for free fields must satisfy the dispersion relation (1) dictated by the κ-deformed invariance of space-time. An example of such equations was proposed for a κ-deformed scalar field [2],

$$\left[\nabla^2 - \left(2\kappa \sin \frac{\partial_0}{2\kappa}\right)^2 - m^2\right] \phi(x) = 0 \,. \tag{2}$$

which can be written as

$$(\nabla^2 - \partial_q^2 - m^2)\phi(x) = 0 \,, \tag{3}$$

by using the deformation parameter $q = (2\kappa)^{-1}$ and the differential operator

$$\partial_q = q^{-1} sin(q\partial_0) \,. \tag{4}$$

Other examples are provided by a κ-deformed Dirac field [2],

$$(i\gamma^j \partial_i - i\gamma^0 \partial_q - m)\psi(x) = 0 \,, \tag{5}$$

where γ^i and γ^0 are usual gamma matrices, and by a κ-deformed Maxwell field [7],

$$\nabla\cdot\mathbf{E}=0\,,\quad \nabla\times\mathbf{E}=-\partial_q\mathbf{B}\,,\quad \nabla\cdot\mathbf{B}=0\,,\quad \nabla\times\mathbf{B}=\partial_q\mathbf{E}\,. \tag{6}$$

Let us consider a κ-deformed scalar field for simplicity and briefly describe its canonical quantization [7], in order to appreciate the peculiarities which follows from the κ-deformation. This field can be described by the action

$$\mathcal{S}(\phi)=\int_\Omega d^4x\,\mathcal{L}(\phi(x),\bar{\partial}\phi(x))\,, \tag{7}$$

where $\bar{\partial}\phi(x)=(\partial_q\phi,\nabla\phi)$ and the Lagrangian density is given by the expression

$$\mathcal{L}(\phi(x),\bar{\partial}\phi(x))=\frac{1}{2}(\partial_q\phi\partial_q\phi-\nabla\phi\cdot\nabla\phi)-\frac{1}{2}m^2\phi^2\,, \tag{8}$$

which differs from the one proposed by Lukierski, Nowicki and Ruegg [2] by surface terms. The action principle requires that the virtual variations of (7) depends only on surface terms and from this principle we obtain the equations of motion (3) by making use of the following lemma

$$\Upsilon\partial_q\Xi=\partial_0\left[-\frac{1}{q}\sum_{n=0}^{\infty}(-1)^n\frac{q^{2n+1}}{(2n+1)!}\sum_{p=1}^{2n+1}(-1)^p(\partial_0^{2n+1-p}\Upsilon\partial_0^{p-1}\Xi)\right]-(\partial_q\Upsilon)\Xi\,, \tag{9}$$

where Υ and Ξ are arbitrary functions of space-time.

Since the κ-deformed Lagrangian density (8) does not depend on space-time explicitly, we can use Noether's theorem to obtain a conserved energy momentum tensor [7]. Here we are interested only in the κ-deformed energy that follows from this tensor, namely

$$P^0=\int d^3\mathbf{x}\,(\Pi^0\partial^0\phi-\mathcal{L}) \tag{10}$$

where we have used the differential operator

$$\Pi^0=-\frac{1}{q}\sum_{n=0}^{\infty}(-1)^n\frac{q^{2n+1}}{(2n+1)!}\sum_{p=1}^{2n+1}(-1)^p\partial_0^{2n+1-p}\frac{\partial\mathcal{L}}{\partial(\partial_q\phi)}\partial_0^{p-1}\,. \tag{11}$$

The quantization of this κ-deformed field can be done in the usual two steps. In the first we promote the field $\phi(x)$ to an operator

$$\phi(x)=\int d^3\mathbf{p}\,\eta(\mathbf{p})\left[a(\mathbf{p})\,e^{-ip\cdot x}+a^\dagger(\mathbf{p})\,e^{ip\cdot x}\right]\,, \tag{12}$$

where $\eta(\mathbf{p})$ is a normalization factor, the plane waves satisfy the equations of motion (3) by obeying the dispersion relation that follows from (1),

$$p^0=\omega(\mathbf{p})=\frac{1}{q}\sinh\left(q\sqrt{\mathbf{p}^2+m^2}\right)\,, \tag{13}$$

and the amplitudes $a(\mathbf{p})$ and $a^\dagger(\mathbf{p})$ are operators obeying the canonical commutation relations

$$[a(\mathbf{p}),a(\mathbf{p}')]=[a^\dagger(\mathbf{p}),a^\dagger(\mathbf{p}')]=0\,,\quad [a(\mathbf{p}),a^\dagger(\mathbf{p}')]=\delta(\mathbf{p}-\mathbf{p}')\,. \tag{14}$$

In the second step of quantization we impose on the field the Heisenberg equation of motion

$$\partial^0\phi = i[P^0,\phi]\ , \tag{15}$$

where P^0 is the operator obtained by quantization of the κ-deformed energy (10).

From the form of equations (14) and (15) the normalization factor in (12) is fixed as $\eta(\mathbf{p}) = [2(2\pi)^3\sinh(2q\omega(\mathbf{p}))/2q]^{-1/2}$. The consistency of the quantization procedure can be verified by applying to both sides of the Heisenberg equations (15) the differential operator $\partial_q^\flat$ obtained from the factorization $\partial_q = \partial_q^\flat\,\partial_0$ of the differential operator (4).

By substituting the expansion (12) into (10) and using the commutation relations (14) we obtain for the κ-deformed energy the expression

$$P^0 = \int d^3\mathbf{p}\,\omega(\mathbf{p})\left(\frac{1}{2}+a^\dagger(\mathbf{p})a(\mathbf{p})\right)\ , \tag{16}$$

where $\omega(\mathbf{p})$ is given by the κ-deformed dispersion relation (13). From this expression we may define the vacuum energy of the κ-deformed field (in box normalization) as the quantity

$$E_0 = \sum_{\mathbf{p}}\frac{1}{2}\omega(\mathbf{p}) = \sum_{\mathbf{p}}\frac{1}{2q}\mathrm{senh}^{-1}(q\sqrt{\mathbf{p}^2+m^2})\ . \tag{17}$$

The simplicity of the result (17) is not only welcome but also remarkable if we follow all the calculations which leads to it. For the κ-deformed electromagnetic field it can be derived an analog expression including the sum over polarizations. Such an expression can be used to obtain the Casimir energy in presence of conducting parallel plates [8] (for a review on the subject of Casimir effect see. *e.g.*, [9, 10]). As a matter fact, the sum over half frequencies expression was assumed to be true in order to obtain the Casimir energy for the κ-deformed electromagnetic field [11, 12] and the result can be written as [12]

$$\mathcal{E}_q(a) = -\frac{-\ell^2}{4\pi^2a^3}\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^{a/q}dy\left(y+\frac{1}{2n}\right)\frac{e^{-2ny}}{\sqrt{1-(qy/a)^2}}\ , \tag{18}$$

where a is the separation between the plates and ℓ the side of each plate, which is assumed to be a square with $\ell >> a$.

From the field expression (12) and the commutation relations (14) we obtain

$$[\phi(x),\phi(x')] = i\Delta_q(x-x')\ , \tag{19}$$

where

$$\Delta_q(x-x') = \frac{-i}{(2\pi^3)}\int\frac{d^3\mathbf{p}}{\sinh 2q\omega(\mathbf{p})/q}\left[e^{-ip\cdot(x-x')}-e^{ip\cdot(x-x')}\right]\ . \tag{20}$$

is the κ-deformed Pauli-Jordan function proposed by Lukierski, Nowicki and Ruegg [2]. From (19) we obtain

$$[\phi(\mathbf{x},t),\pi_{2q}(\mathbf{x}',t)] = i\delta(\mathbf{x}-\mathbf{x}') \tag{21}$$

where

$$\pi_{2q}(x) = \partial_{2q}\phi(x)\ . \tag{22}$$

Several remarks are in order concerning the above formalism. First of all, we should notice that the introduction of a scale parameter into the space-time symmetries through κ-deformation leads us to a field described by a Lagrangian density (8) of infinite order in time derivative, due to the presence of the infinite order differential operator (4). The corresponding equation of motion (3) is also of infinite order in time derivative and cannot be used to construct solution from a finite set of initial data. In this respect, however, two facts must be taken into consideration. The first is that the equation of motion (3) fulfills its fundamental purpose of defining the set of all possible motions of the field. It is given by the kernel of the operator $\nabla^2 - \partial_q^2 - m^2$ in the space of smooth functions. By substituting any smooth function in equation (3) it can be unequivocally determined if it belongs or not to this kernel. We could determine that plane waves with dispersion relation (13) belongs to this kernel and so all field operators (12) of rapid decrease at infinity. The second fact is that a drastic change of scale may give rise to a change in the deterministic features of theories, as when we change 8 orders of magnitude in length from the realm of classical mechanics to the scale of quantum mechanics. If κ-deformation is supposed to describe physics at very small scale, say the Planck scale, it is plausible to expect different deterministic properties in the theory due to a change of about 17 orders of magnitude from the scale of usual quantum field theories to this scale. Another consequence of the infinite order of the theory is that surface terms in the action (7) depend on the values of all time derivatives of the field.

Another important peculiarity of the κ-deformed theory is that different operators play the role of the field conjugate to ϕ, depending on the part of the theory into consideration. In the equal time canonical commutation relation (21) the conjugate field is given by the field operator π_{2q} defined by (22), while in the expression (10) is given by the differential operator Π^0 with field operators as coefficients, as defined in (11). We should add that the Legendre transform conjugate variable $\pi_q(x) = \partial\mathcal{L}/\partial(\partial_q\phi(x))$ does not satisfy the equal time canonical commutation relation, $[\phi(\mathbf{x},t),\pi_q(\mathbf{x}')] \neq \delta(\mathbf{x}-\mathbf{x}')$), nor the Legendre transform of the Lagrangian will give the operator P^0 obtained in (10) from Noether's theorem. However, all the different operators playing some of the roles of a conjugate field have the same limit when the deformation disappears, $\lim_{q\to 0}\pi_{2q} = \lim_{q\to 0}\pi_q = \lim_{q\to 0}\Pi^0 = \partial_0\phi$. After all, it is important to notice that a procedure along the essential lines of canonical quantization can be followed to obtain a quantized κ-deformed field.

Let us now add to the above formalism a new result concerning its Green's functions. We may first verify that by requiring from vacuum state the property $a(\mathbf{p})|0\rangle = 0$ the propagator $\langle 0|T(\phi(x)\phi(x'))|0\rangle$ is not a Green function of the theory. On the other hand, the amplitude $\langle 0|T(\chi(x)\phi(x'))|0\rangle$, where

$$\chi(x) = q\partial_0 \cot(q\partial_0)\phi(x) \,, \tag{23}$$

is a Green's function of the theory

$$(\nabla^2 - \partial_q^2 - m^2)\langle 0|T(\chi(x)\phi(x'))|0\rangle = \delta(x-x') \,. \tag{24}$$

But none of those functions is the κ-causal Green's function $G_q(x,x')$, defined as the inverse of the differential operator $\nabla^2 - \partial_q^2 - m^2 + i\varepsilon$, where ε is a small positive quantity.

Let us now turn to some results that can be obtained by functional quantization of a κ-deformed theory. The effective action for the κ-deformed free field is given in Schwinger proper-time representation by [13]

$$\mathcal{W} = -\frac{1}{2}\int_0^\infty \frac{ds}{s} Tr\, e^{-isH} \;, \tag{25}$$

where H is the proper-time Hamiltonian determined from the Casimir invariant (1) and given by

$$H = \mathbf{P}^2 - \left(2\kappa \sinh \frac{P^0}{2\kappa}\right)^2 + m^2 \;. \tag{26}$$

From these expressions it can be shown that (in box normalization)

$$\mathcal{W} = -T \sum_{\mathbf{p}} \frac{1}{2}\omega(\mathbf{p}) + i\frac{-T}{\pi\kappa}\sum_{\mathbf{p}}\frac{1}{4}\omega(\mathbf{p})^2 \;, \tag{27}$$

where T is a large elapsed time from remote past to distant future and the frequencies $\omega(\mathbf{p})$ are given by the κ-deformed dispersion relation (13). The expression of the real part of this effective action as proportional to the vacuum energy (17) is an expected result, while the imaginary part is a contribution peculiar from the κ-deformation. In free space both real and imaginary part of this κ-deformed effective action can be subtracted out as spurious contribution. However, in the presence of boundary conditions they give rise to observables. From the real part we obtain the Casimir energy related to the given boundary conditions and from the imaginary part we get the creation rate of field excitations of the field in the presence of the boundary conditions. Both quantities have been obtained directly from the proper-time representation (25) of the effective action [14] in the case of the κ-deformed scalar field under Dirichlet boundary conditions on two large parallel squares of side ℓ and separation a. They are given by

$$\mathcal{W} = \frac{T\, a\ell^2}{16\pi^2 a^4}\sum_{n=1}^{\infty}\int_0^\infty d\sigma\,\sigma\, e^{-n^2\sigma-(2\kappa^2+m^2)a^2/\sigma}\sqrt{\frac{4(a\kappa)^2}{\pi\sigma}}\left[\pi I_0(2(a\kappa)^2/\sigma) + iK_0(2(a\kappa)^2/\sigma)\right] \;. \tag{28}$$

In the limit $\kappa \to \infty$, the real part of this expression gives the corresponding Casimir energy of the usual scalar field. In the same limit the imaginary part goes to zero. On the other hand, in the limit $a \to \infty$ both real and imaginary parts go to zero, *i.e.*, the Casimir energy and the excitation creation rate go to zero when the confining boundary conditions are withdrawn. The result (28) can be applied to cosmological considerations [14]. If we consider an universe with dimensions of the order of a in its early stages, the imaginary part of (28) provides a mechanism of creation of matter (or radiation in the case $m = 0$) out of the confined κ-deformed space. The greater the product $a\kappa$, the smaller the creation rate. The mechanism of creation becomes negligible for a size a large enough. We may also consider the possibility that the κ-deformation also decreases with time in order to eliminate a significant creation rate.

The result for the effective action (28) shows that a κ-deformed field presents a mechanism of creation of excitations which does not appear in the canonical quantization in a obvious way, and which prevents us of postulating the condition $a(\mathbf{p})|0\rangle = 0$ when boundary conditions are present. As a matter of fact, by taking the vacuum expectation value of the operator P^0 in (16) we should obtain $-\mathcal{W}/T$ with $\mathcal{W}$ given by the expression (28). In order to obtain from (16) the imaginary part that appears in this expression we would need a very precise prescription for the vector $a(\mathbf{p})|0\rangle$ to be added by hand to the canonical quantization method. This consideration favors the functional method of quantization in face of the canonical for the case of κ-deformed fields. By adding to this observation the peculiarities of the canonical quantization discussed above we may conclude that further investigations are necessary to know precisely how to quantize κ-deformed field theories.

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