PoS PROCEEDINGS OF SCIENCE

On Solitons, Non-Linear Sigma-Models, and Two-Dimensional Gravity

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Some interesting inter-connections between solitons, non-linear sigma-models, and gravity (in two and four dimensions) are discussed. Certain sigma-models and non-constant scalar curvature metrics are constructed from generalized solitons. Speculation is presented whether such metrics can be transformed (by a suitable change of coordinates) to black hole metrics.

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1. Introduction

Various connections between non-linear sigma-models and gravity have been the subject of discussion for some forty years or so; compare [1,2,3,8,10,11,12,13]. Recent discussions have involved links between solitons and gravity [5,6,7,16,17]. We extend these discussions here, where a link between solitons, sigma-models, and two-dimensional Jackiw-Teitelboim gravity is presented. Also presented is a construction of sigma-models (specifically maps of the plane to the 2-sphere), given solitons of a generalized type, and we construct corresponding metrics that we propose should be of interest for more general theories of two-dimensional dilaton gravity. Some background material on constant curvature metrics and sine-Gordon solitons is included before generalized sine-Gordon equations are considered.

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2. Constant curvature metrics, sine-Gordon solitons, and two-dimensional gravity

The connection between constant curvature metrics and solutions of sine-Gordon equations is reviewed here, with the introduction of some notation. For a pseudo Riemannian manifold (M,g) with local expression

$$ds^2 = \sum_{i,j=1}^{m=dimM} g_{ij} dx_i dx_j \tag{2.1}$$

of the metric g, we shall observe the following sign convention for the curvature tensor R^l_{ijk} and the scalar curvature $R = R(g)$ of g, where $g^{-1} = [g^{ij}]$:

$$\begin{aligned} R^l_{ijk} &= \frac{\partial \Gamma^l_{jk}}{\partial x_i} - \frac{\partial \Gamma^l_{ik}}{\partial x_j} + \sum_{p=1}^{m} [\Gamma^l_{ip}\Gamma^p_{jk} - \Gamma^l_{jp}\Gamma^p_{ik}], \\ R &= \sum_{i,j=1}^{m} g^{ij} R_{ij} . \end{aligned} \tag{2.2}$$

Here

$$R_{ij} = \sum_{l=1}^{m} R^l_{ilj} \, , \Gamma^k_{ij} = \frac{1}{2} \sum_{l=1}^{m} g^{lk} [\frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l}] \tag{2.3}$$

are the Ricci tensor and Christoffel symbols, respectively, of g. The Gaussian curvature K of g is $K = -R/2$. We will have an interest in the particular case when M is two-dimensional: $m = 2$. In this case one always has the formula $R_{ij} = (R/2)g_{ij}$. That is, the Einstein vacuum equations $R_{ij} - \frac{R}{2} g_{ij} + \Lambda g_{ij} = 0$ *automatically hold* for a vanishing cosmological constant Λ. These equations consequently are of less interest and one considers instead the *non-trivial* Einstein equation (in two dimensions)

$$R(g) = A(\text{a constant}), \tag{2.4}$$

due to Jackiw-Teitelboim (J-T) [9,15]. Equation (2.4) is derived from the J-T action integral

$$I_{\text{J-T}}(\tau,g) = \frac{1}{2G}\int_M \sqrt{|detg|}\,dx_1\,dx_2\,(A-R(g))\tau \tag{2.5}$$

by variation of a scalar field $\tau(x_1,x_2)$(called a *dilation*). The example

$$\tau(x,t) \stackrel{def}{=} \sqrt{1+v^2}\,\text{sech}[\frac{m(x-vt)}{\sqrt{1+v^2}}] \tag{2.6}$$

(with $(x_1,x_2)=(x,\tau)$) appears in section 3 of [6].

Given A , solutions g of equation (2.4) can be obtained on the basis of a well-known observation, where we denote the coordinates (x_1,x_2) by (x,y): For a function $f(x,y)$, the metric g defined by

$$ds^2 = \cos^2(f(x,y))dx^2 + \sin^2(f(x,y))dy^2 \tag{2.7}$$

(with $g_{12}=g_{21}=0$) has scalar curvature

$$R(g) = 4(f_{xx}-f_{yy})/\sin 2f. \tag{2.8}$$

This follows by (2.3),(2.4), or more directly by the formula

$$\begin{aligned} R = \frac{2}{g_{11}g_{22}}R_{1212} &\stackrel{i.e.}{=} \frac{2}{g_{11}g_{22}}[\frac{1}{2}\frac{\partial^2 g_{11}}{\partial y^2}+\frac{1}{2}\frac{\partial^2 g_{22}}{\partial x^2}-\frac{1}{4g_{11}}(\frac{\partial g_{11}}{\partial y})^2 \\ &-\frac{1}{4g_{22}}(\frac{\partial g_{22}}{\partial x})^2-\frac{1}{4g_{11}}\frac{\partial g_{11}}{\partial x}\frac{\partial g_{22}}{\partial x}-\frac{1}{4g_{22}}\frac{\partial g_{11}}{\partial y}\frac{\partial g_{22}}{\partial y}] \end{aligned} \tag{2.9}$$

of Gauss (for any two-dimensional metric with $g_{12}=g_{21}=0$) [14]. Equation (2.8) means that g in (2.7) is a solution of the Einstein equation (2.4) $\Longleftrightarrow$ $f(x,y)$ is a solution of the sine-Gordon equation

$$f_{xx}-f_{yy} = \frac{A}{4}\sin 2f. \tag{2.10}$$

Examples of solutions $f(x,y)$ of equation (2.10) (besides the trivial constant solutions $n\pi, n\in\mathbb{Z}=$ the ring of integers) are the soliton (solid wave) solutions:

1. $f(x,y)=2\arctan[\exp(a(x-vy))]$ where $a=(1-v^2)^{-1/2}, A=2$-$i.e.K=-1$.

2. $f(x,y)=2\arctan[\sinh(avy)/v\cosh(ax)]$ for a, A in example 1. This is a *soliton-antisoliton* (or kink-antikink) soliton.

3. $f(x,y)=2\arctan[a\sin(vy)/v\cosh(ax)]$ where $a=(1-v^2)^{1/2}, A=2$. This is a *breather* solution.

Of interest as well is a Euclidean version

$$\Delta u \stackrel{def}{=} \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial t^2} = m^2\sin u(x,t) \tag{2.11}$$

of the sine-Gordon equation. Here one considers the metric

$$ds^2 = \cos^2(\frac{u}{2}(x,t))dx^2 - \sin^2(\frac{u}{2}(x,t))dt^2 \tag{2.12}$$

(with coordinates $(x_1,x_2) = (x,t)$), in contrast to the one in (2.7). Using the Gauss formula (2.9) one has, similarly, that $R = 2\Delta/(\sin u)$, and therefore this metric solves the Einstein equation (2.4) $R = 2m^2 \iff u$ satisfies (2.11). Here m is any positive real number. For $v > 0$ define

$$a = a(v) = (1+v^2)^{\frac{1}{2}}, \rho(x,t) = \frac{m(x-vt)}{a(v)}, \beta(x,t) = \frac{m(vx+t)}{a(v)}. \tag{2.13}$$

Then the dilation $\tau(x,t)$ in (2.6) is expressed as

$$\tau(x,t) = a(v)\mathrm{sech}\rho(x,t), \tag{2.14}$$

and

$$u^{\pm}(x,t) \overset{def}{=} 4\arctan[\exp(\pm\rho(x,t))] \tag{2.15}$$

are the soliton solutions of (2.11) analogous to the solution $f(x,y) = 2\arctan[\exp a(x-vy)]$ of (2.10) in example 1. Moreover the functions $\Phi^{\pm} : \mathbb{R}^2 \to \mathbb{S}^2$ from the plane to the unit 2-sphere given by

$$\Phi^{\pm} \overset{def}{=} (\cos\beta\sin\alpha^{\pm}, \sin\beta\sin\alpha^{\pm}, \cos\alpha^{\pm}) \tag{2.16}$$

for $\alpha^{\pm} \overset{def}{=} u^{\pm}/2$ are non linear $\sigma-$models -*i.e.* they are *harmonic maps* in the sense of J. Eells and J. Sampson[4]; see section 3. One has that

$$\sin\alpha^{\pm} \overset{def}{=} \sin\frac{u^{\pm}}{2} = \mathrm{sech}\rho,\ \cos\alpha^{\pm} = \mp\tanh\rho. \tag{2.17}$$

Consequently we can also write

$$\Phi^{\pm} = \frac{1}{a(v)}(\tau\cos\beta, \tau\sin\beta, \mp a(v)\tanh\rho). \tag{2.18}$$

Equations (2.16), (2.18) connect solitons $u^{\pm}$, non-linear $\sigma-$models $\Phi^{\pm}$, and two-dimensional gravity via the dilation τ, where moreover the metric g in (2.12) for $u = u^{\pm}$ satisfies the two-dimensional Einstein-Jackiw-Teitelboim field equation $R(g) = 2m^2$ (by the remark following (2.12)) and is known to transform to the black hole metric

$$ds^2 = -(m^2r^2 - v^2)dT^2 + \frac{dr^2}{(m^2r^2 - v^2)} \tag{2.19}$$

by a suitable change of variables $(x,t) \to (T,r)$. The explicit transformation $\Theta(x,t) = (\theta_1(x,t), \theta_2(x,t)) = (T,r)$ of the metric (2.12) to the metric (2.19) is given, in fact, by

$$\begin{aligned}\theta_1(x,t) &= \frac{-1}{2mv}\log[\frac{a(v)\tanh\rho(x,t)+1}{a(v)\tanh\rho(x,t)-1}] + \frac{x}{v},\\ \theta_2(x,t) &= \tau(x,t)/m,\end{aligned} \tag{2.20}$$

which implements an observation of J. Gegenberg and G. Kunstatter [5,6], as discovered in [16,17].

3. A generalization of the sigma-models $\Phi^{\pm}$ in equation (2.16)

The main point of this section is to generalize the construction of the maps $\Phi^{\pm} : \mathbb{R}^2 \to \mathbb{S}^2$ in (2.16) in a way to produce new sigma-models. As the $\Phi^{\pm}$ were constructed via the solitons $u^{\pm}$ in (2.15), we shall seek first an appropriate replacement of these functions. We also consider the metric in (2.12) where u is not necessarily a solution of the sine-Gordon equation, and the implication of such a metric for gravity.

For the sake of completeness, we define a harmonic map (or non-linear sigma-model) $\Phi : (M,g) \to (N,h)$ of pseudo Riemannian manifolds. We proceed locally although a global, coordinate - independent definition is also available [4]. Let $(U, \phi = (x_1,\ldots,x_m))$, $(V, \psi = (y_1,\ldots,y_n))$ be local coordinate systems on M,N with $U \subset \Phi^{-1}(V)$ so that one can consider the j^{th} coordinate functions $\Phi^j \stackrel{def}{=} y_j \circ \Phi \circ \phi^{-1} (1 \leq j \leq n)$ relative to these systems. We assume that Φ is a smooth map. Write $\partial_j = \frac{\partial}{\partial x_j}$ and let Δ_g denote the Laplace-Beltrami operator of g:

$$\Delta_g \stackrel{def}{=} \frac{1}{\sqrt{|detg|}} \sum_{i,j=1}^{m} \partial_i [\sqrt{|detg|} g^{ij} \partial_j] \tag{3.1}$$

on U. If Γ^k_{ij} are the Christoffel symbols of h (see (2.3), with g there replaced by h) then the *non-linear Laplacians* $\tilde{\Delta}_s (1 \leq s \leq n)$ are defined to act on Φ by

$$(\tilde{\Delta}_s \Phi)(p) \stackrel{def}{=} \sum_{i,j=1}^{m} (g^{ij} \circ \phi^{-1}) \sum_{k,r=1}^{n} \partial_i \Phi^k \partial_j \Phi^r |_{\phi(p)} \Gamma^s_{kr}(\Phi(p)) + \Delta_g \Phi^s |_{\phi(p)} \tag{3.2}$$

for $p \in U$. Φ is *harmonic* if it satisfies the system of equations

$$(\tilde{\Delta}_s \Phi^s) = 0, (1 \leq s \leq n = dimN). \tag{3.3}$$

The field equation (3.3) can be derived by a variational principle where the *energy integral* of Φ is made stationary with respect to Φ. For a Bosonic string, for example, this integral is the Polyakov integral and the equations (3.3) coincide with the equation of the motion of the string - say for M= its two-dimensional world sheet and $N = \mathbb{R}^{26}$, 26 being the critical dimension. If $M \subset \mathbb{R}^1$ is some interval, then Φ is simply a smooth curve in N and the equations (3.3) are the familiar conditions that Φ should be a geodesic. If N is a flat space with vanishing Christoffel symbols Γ^k_{ij} then the conditions (3.3) reduce to the standard conditions for harmonicity. In the case of $M = \mathbb{R}^2$, $N = \mathbb{S}^2$ with their standard Riemannian metrics, one has the following result. Given smooth functions $\alpha, \beta : \mathbb{R}^2 \to \mathbb{R}$, the function $\Phi = \Phi_{\alpha,\beta} : \mathbb{R}^2 \to \mathbb{S}^2$ defined by

$$\Phi_{\alpha,\beta} = (\cos\beta \sin\alpha, \sin\beta \sin\alpha, \cos\alpha) \tag{3.4}$$

is harmonic (-*i.e.* it satisfies conditions (3.3)) if α, β satisfy the conditions

$$\Delta\alpha \stackrel{def}{=} \frac{\partial^2\alpha}{\partial x^2} + \frac{\partial^2\alpha}{\partial t^2} = [(\frac{\partial\beta}{\partial x})^2 + (\frac{\partial\beta}{\partial t})^2] \sin\alpha \cos\alpha,$$
$$(\sin\alpha)\Delta\beta + 2[\frac{\partial\alpha}{\partial x}\frac{\partial\beta}{\partial x} + \frac{\partial\alpha}{\partial t}\frac{\partial\beta}{\partial t}] \cos\alpha = 0. \tag{3.5}$$

For example, $\alpha^{\pm} \stackrel{def}{=} u^{\pm}/2$ and β in (2.13) satisfy the system (3.5), since we have noted that $u^{\pm}$ satisfy the Euclidean sine-Gordon equation (2.11). Hence one can conclude that $\Phi^{\pm} = \Phi_{\alpha^{\pm},\beta}$ in (2.16) are harmonic maps, as asserted in section 2.

A fifth example, which nicely connects non-linear sigma-models and gravity (this time four-dimensional gravity) is obtained by taking $M = \mathbb{R}^2$, $N = \mathbb{R} \times (\mathbb{R} - \{0\}) \times \mathbb{R} \times \mathbb{R}$,

$$g(x,t) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, h(y_1,y_2,y_3,y_4) = \begin{bmatrix} -e^{y_1} & 0 & 0 & e^{y_1} \\ 0 & e^{y_1} & 0 & 0 \\ 0 & 0 & e^{y_1} sh^2 y_2 & 0 \\ e^{y_1} & 0 & 0 & 0 \end{bmatrix} \tag{3.6}$$

where sh denotes the hyperbolic sine. cth similarly will denote the hyperbolic cotangent. The conditions (3.3) here (where $s = 4$) reduce to the following, where we write $\Phi = (\Phi^1, \Phi^2, \Phi^3, \Phi^4)$, $\Delta_g = \frac{\partial}{\partial x^2} - \frac{\partial}{\partial t^2}$ (= the Laplace- Beltrami operator of g):

$$\begin{aligned} \left[\Phi^1_x\right]^2 - \left[\Phi^1_t\right]^2 + \Delta_g \Phi^1 &\stackrel{(i)}{=} 0, \\ \Phi^1_x \Phi^2_x - \Phi^1_t \Phi^2_t - \frac{1}{2}\left[(\Phi^3_x)^2 - (\Phi^3_t)^2\right] sh 2\Phi^2 + \Delta_g \Phi^2 &\stackrel{(ii)}{=} 0, \\ \Phi^1_x \Phi^3_x - \Phi^1_t \Phi^3_t + 2\left[\Phi^2_x \Phi^3_x - \Phi^2_t \Phi^3_t\right] cth \Phi^2 + \Delta_g \Phi^3 &\stackrel{(iii)}{=} 0, \\ \frac{1}{2}\left[(\Phi^1_x)^2 - (\Phi^1_t)^2\right] - \frac{1}{2}\left[(\Phi^2_x)^2 - (\Phi^2_t)^2\right] - \frac{1}{2}\left[(\Phi^3_x)^2 - (\Phi^3_t)^2\right] sh^2 \Phi^2 + \Delta_g \Phi^4 &\stackrel{(iv)}{=} 0. \end{aligned} \tag{3.7}$$

These equations follow by a direct computation of the Christoffel symbols of h in (3.6); see Appendix 1. On the other hand, the conditions $(i), (ii), (iii), (iv)$ (for $\Phi : M \to N$ to be a sigma-model) are *exactly* the Einstein gravitational equations for a 4-dimensional *plane-symmetric* space-time. Thus one has another beautiful connection between non-linear sigma-models and gravitation. This latter one is due to S.Chervon and A. Muslimov [1]; also see [2,3,13].

The key to generalizing the functions $u^{\pm}$ in (2.15), and hence the functions $\Phi^{\pm}$, is the following very simple observation: The pair (ρ, β) in (2.13) satisfies the Cauchy-Riemann (C-R) equations: $\rho_x = \frac{m}{a} = \beta_t, \rho_t = \frac{-mv}{a} = -\beta_x$. Thus ρ and β are harmonic conjugates. This observation motivates us now to choose ρ to be *any* harmonic function on the plane $\mathbb{R}^2$: $\Delta\rho = \frac{\partial^2\rho}{\partial x^2} + \frac{\partial^2\rho}{\partial t^2} = 0$. Since $\mathbb{R}^2$ is simply connected we now choose β to be a harmonic conjugate of ρ: $\rho + \sqrt{-1}\beta$ is an analytic function. Motivated by (2.15) and the definition $\alpha^{\pm} = u^{\pm}/2$, we define

$$u(x,t) \stackrel{def}{=} 4\arctan(\exp\rho), \alpha \stackrel{def}{=} \frac{u}{2}. \tag{3.8}$$

One has (compare (2.17))

$$\begin{aligned} \sin\alpha = \operatorname{sech}\rho \quad &, \quad \cos\alpha = -\tanh\rho, \\ \alpha_x = \frac{1}{2}u_x = \rho_x \operatorname{sech}\rho \quad &, \quad \alpha_t = \frac{1}{2}u_t = \rho_t \operatorname{sech}\rho, \\ \Delta u = 2(\Delta\rho)\operatorname{sech}\rho \; - \; & 2(\operatorname{sech}\rho\tanh\rho)(\rho_x^2 + \rho_t^2) \end{aligned} \tag{3.9}$$

where $\Delta\rho = 0$, by hypothesis. Therefore $\Delta\alpha = (\sin\alpha\cos\alpha)(\rho_x^2+\rho_t^2) = (\sin\alpha\cos\alpha)(\beta_t^2+\beta_x^2)$ (by the C-R equations), which is the first equation in (3.5). The second equation there also holds since $\Delta\beta = 0$ (by definition of β), and since $\alpha_x\beta_x+\alpha_t\beta_t \overset{def}{=} \frac{1}{2}(\rho_x\beta_x+\rho_t\beta_t) = \frac{1}{2}[\rho_x(-\rho_t)+\rho_t\rho_x]$ (again by the C-R equations) $= 0$. In summary we have therefore shown the following.

Theorem 1. Let $\rho(x,t)$ be any harmonic function on $\mathbb{R}^2 : \Delta\rho = \rho_{xx}+\rho_{tt} = 0$. Let $\beta(x,t)$ be a harmonic conjugate of $\rho(x,t)$. Define u and α by (3.8). The α and β are solutions of the system of equations in (3.5), and hence the function $\Phi_{\alpha\beta} : \mathbb{R}^2 \to \mathbb{S}^2$ defined in (3.4) is non-linear sigma-model - *i.e.* $\Phi_{\alpha\beta}$ satisfies the system of equations (3.3). Also u satisfies the generalized type of sine-Gordon equation

$$\Delta u = (\rho_x^2+\rho_t^2)\sin u \tag{3.10}$$

(by (3.9)), which contrasts equation (2.11).

Consider the metric in (2.12) where we now take u there to be the function in (3.8) for ρ in Theorem 1. Denote this metric by g_ρ, which is a generalized type of soliton metric, given equation (3.10). By the remark following (2.12) it scalar curvature is given by $2\Delta u/(\sin u)$, which by equation (3.10) equals $2(\rho_x^2+\rho_t^2) : R(g_\rho) = 2(\rho_x^2+\rho_t^2)$, which generally is non -constant -*i.e.* g_ρ generally will not solve equation (2.4). One can determine all harmonic conjugate pairs (ρ,β) for which $R(g_\rho)$ is a constant. Such pairs are given by $\rho(x,t) \overset{(i)}{=} ax-bt+c, \beta(x,t) \overset{(ii)}{=} bx+at+d$ for suitable real numbers a,b,c,d (which is consistent with the pair (ρ,β) given in (2.31)). To see this, let $f = \rho+i\beta$ be the corresponding analytic function. Then $f' = \rho_x+i\beta_x = \rho_x-i\rho_t$, by C-R, $\Rightarrow R(g_\rho) = 2|f'|^2$. In particular if $R(g_\rho)$ is a constant then $|f'|$ is a constant, and since $f'(z)$ is also analytic one may conclude that $f'(z)$ is a constant: $\rho_x+i\beta_x = a+ib \Rightarrow \rho(x,t) = ax+c(t), \beta(x,t) = bx+d(t)$, where by C-R, $a = \rho_x = \beta_t = d'(t), c'(t) = \rho_t = -\beta_x = -b \Rightarrow d(t) = at+d, c(t) = -bt+c$, which proves (i) and (ii).

Given the metric g_ρ, an obvious and very interesting question arises: Can one construct a transformation of variables $\Theta_\rho : \mathbb{R}^2 \to \mathbb{R}^2, (x,t) \to (T,r)$, under which g_ρ goes (perhaps) to a black hole metric G_ρ (as we did in (2.20) in the constant curvature set-up $R = 2m^2$)? One would like G_ρ to assume the form

$$ds^2 = A(r)dT^2 - \frac{dr^2}{A(r)}, \tag{3.11}$$

for example; compare (2.19). The latter metric has scalar curvature $-A''(r)$(by(2.9)). In this more general setting we replace the J-T action integral given in (2.5) by

$$I(g,\tau) = \frac{1}{2G}\int_M \sqrt{|detg|}\,dx_1\,dx_2[V\circ\tau - R(g)\tau] \tag{3.12}$$

with equations of motion

$$\begin{aligned} R(g) &= \frac{dV}{dx_2}\circ\tau \quad \text{(varying } \tau\text{)}, \\ \Delta_g\tau+V\circ\tau &= 0 \quad \text{(varying } g\text{)}, \end{aligned} \tag{3.13}$$

where V is a function of x_2 only and Δ_g is given by (3.1); see [7]. Hopefully, future work will lead to a construction of Θ_ρ. We note in fact that for the coordinates $(x_1, x_2) = (T, r)$, the metric G_ρ in (3.11) and the dilation $\tau(T,r) \overset{def}{=} r$ do provide a solution of the field equations (3.13), for $A'(r) = V(r)$. This follows from (3.1) which for $g = G_\rho$ gives

$$\Delta_{G_\rho} = \frac{1}{A(r)}\frac{\partial^2}{\partial T^2} - A(r)\frac{\partial^2}{\partial r^2} - A'(r)\frac{\partial}{\partial r}. \tag{3.14}$$

Thus indeed $V'(\tau(T,r)) = A''(r) = -R(G_\rho)(T,r)$ (as noted in the line following (3.11)), which is the first equation in (3.13), and $(\Delta_{G_\rho}\tau)(T,r) + V(\tau(T,r)) = -A'(r) + A'(r) = 0$, which is the second equation in (3.13).

Note also that the function

$$u_2(x,t) = 4\arctan\left[\frac{v}{\sqrt{1+v^2}}(\sinh\sqrt{1+v^2}mx)\sec vmt\right] \tag{3.15}$$

is a solution of the Euclidean sine-Gordon equation $u_{xx} + u_{tt} = m^2 \sin u$ in (2.11). This can be verified, for example, by a simple Maple program.

Given the solution (3.15), we can form the corresponding soliton metric g_{u_2} in (2.12) which , in contrast to g_ρ, has constant curvature $R = 2m^2$ (again by the formula $R = 2\Delta/(\sin u)$ following (2.12)). Similar to the question posed for the metric g_ρ, it is meaningful to inquire whether one can construct a transformation Φ_{u_2} (as was done in (2.20) for the solitons in (2.15)) that realizes g_{u_2} as a black hole metric. This is a question that my student, Miss S. Beheshti, is considering. The solution $u_2(x,t)$ is also called a kink-antikink solution. It describes a collision between a kink soliton and an antikink soliton.

Appendix 1

For the sake of completeness of the discussion in Sec. 3, we list the values of all the Christoffel symbols of the metric h in (3.6). For $y = (y_1, y_2, y_3, y_4) \in \mathbb{R} \times (\mathbb{R} - \{0\}) \times \mathbb{R} \times \mathbb{R}$, $\Gamma^1_{11}(y) = 1$, $\Gamma^2_{12}(y) = \frac{1}{2}$, $\Gamma^2_{33}(y) = (-\sinh y_2)\cdot\cosh y_2$, $\Gamma^3_{13}(y) = \frac{1}{2}$, $\Gamma^3_{23}(y) = \frac{\cosh y_2}{\sinh y_2}$, $\Gamma^4_{11}(y) = \frac{1}{2}$, $\Gamma^4_{22}(y) = -\frac{1}{2}$, $\Gamma^4_{33}(y) = \frac{-(\sinh y_2)^2}{2}$. All other symbols are zero ; of course one has the symmetry $\Gamma^k_{ij} = \Gamma^k_{ji}$.

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