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Deriving SUSY

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Abstract: The consequences of certain simple assumptions like smoothness of ground state properties and vanishing of the vacuum energy (at least perturbatively) are explored. It would be interesting from the point of view of building realistic theories to obtain these properties without supersymmetry. Here we show, however, at least in some quantum mechanical models, that these simple assumptions lead to supersymmetric theories.

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1. Introduction

One may wonder why it is so that the energy spectrum of nature – locally, i.e. ignoring gravity – seems to have a bottom, but no top. Having in mind that there are many parameters – coupling constants – which are so far not understood in the sense that we do not have any theory telling why they should just be what they are, one may ask: If we varied these parameters/couplings , would the bottom perhaps disappear? Would the energy density of the ground state – essentially the cosmological constant – remain small?

It is of course well known that SUSY theories give zero energy for the ground state and have been therefore considered as the possible key to the solution of the small cosmological constant problem (see [1] for a recent review). SUSY was also shown to have very simple smoothness properties (see e.g. [2]). However it is not obvious that there are no non-supersymmetric field theories with such properties. In fact, that would be even desirable from the point of view of building realistic models. Recently there was such an attempt. More precisely, a nonsupersymmetric string theory was presented which was argued to have vanishing cosmological constant [3] (see also [4, 5]). However, the claims in [3] were criticised by Iengo and Zhu [6].

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[†]This talk was scheduled but not presented due the travel problems.

In order to understand if non-SUSY theories with such properties exist, here we propose to investigate the opposite. We want to start from some simple assumptions, like vanishing of vacuum energy and/or certain smoothness properties of ground state, and to consider which interactions are allowed with these requirements. We shall see that such assumptions will (at least in cases considered in this paper) lead us from bosonic theories to SUSY theories with fermion degrees of freedom.

For a simplest example of the above ideas let us consider the harmonic oscillator. The ground state energy is $E_g = |\hbar\omega|/2$, which is not analytic when $\hbar$ (or ω) is passing zero. The "minimal" way to make it smooth is to double the number of deegres of freedom and add to Hamiltonian a term $\hbar\omega\sigma_z/2$. What one obtains is exactly N=2 SUSY harmonic oscillator. This argument perturbatively generalises to arbitrary one-dimensional QM model.

Another example is obtained when one considers bosonic D-dimensional QM system with several classical vacua of vanishing energy. Trying to keep perturbatively this property in quantum case by changing the Hamiltonian in a "minimal" way for a *generic* choice of coupling constants, one is led to SUSY QM [8].

In the main part we take yet another route. We shall start from QM on compact configuration spaces and assume that the classical limit shall work (perturbatively) even when formally continuing to negative $\hbar$. Since wave packets representing classical states tend to jump around under $\hbar$-changing-sign continuation, we are suggested to identify in the classical interpretation the startpoint and the endpoint for such jumps. Thereby strong classical symmetry between different points in configuration space is to be imposed to uphold the good classical limit and that is how both an effective fermionic degree of freedom and SUSY comes in, unavoidably.

2. "Als ob" fermions from bosons

We consider particle moving in D-dimensional compact Riemann space with the Hamiltonian given with

$$H = -\hbar^2\nabla^2 + V(q)$$

where $q = (q_1, \ldots, q_D)$ are coordinates, and ∇ the covariant derivative. The potential $V(q)$ can be written [9] in the form of Riccati equation

$$V(q) - E_g = (\nabla W(q))^2 - \hbar\nabla^2 W(q) \tag{2.1}$$

where E_g is the ground state energy, and $W(q)$ is connected to the ground state wave function $\psi_g(q)$ with $W(q) = -\hbar \ln \psi_g(q)$. We want again for our system to have "smooth" classical limit, so we take $V(q)$, E_q and $W(q)$ to be "expandable" in $\hbar$, e.g.:

$$W(q) = \sum_{n=0}^{\infty} \hbar^n W^{(n)}(q) \tag{2.2}$$

and simmilar for $V(q)$ and E_q. This leads us to two statements proved in [8]:

Statement 1. The classical potential $V^{(0)}(q)$ has at least two equally deep minima, i.e., there exists at least two points q_i for which $V^{(0)}(q_i) - E_g^{(0)} = 0$. More precisely, number of these classical minima is equal to the number of critical points of $W^{(0)}(q)$.

Statement 2. Main concentration of probability for the ground state (measured by $|\psi_g(q)|^2$) will jump from around the global maximum to around the global minimum of $W(q)$ when $\hbar$ is continously passing by $\hbar = 0$.

It is the crux of our "derivation" of (need for) SUSY that we declare:

Such a "jumping" under $\hbar$ passing $\hbar = 0$ (from $\hbar > 0$ to $\hbar < 0$) means that the classical limit is not good (i.e., "smoothness assumptions" are violated)!

Our "solution" to the jumping-of-states-to-different-minimum-of-V-problem is proposed as:

We propose to change the classical configuration space by putting together to one point so many points as are needed to have all the "jumps" for $\hbar \to -\hbar$ occur between original q-points now identified to be interpreted as only one point.

If we want to have classical physics not to distinguish the points to be identified – say we identify $q \to f(q)$ – then at least to classical approximation we must have

1. The map f : configuration space $\to$ configuration space being an isometry for the metric $g_{ab}(q)$ of the kinetic term.
2. $V(f(q)) = V(q)$

We expect that additional variables, introduced to denote different (bosonic) configurations which are classically indistinguishable, will behave as fermionic degrees of freedom, at least locally around classical vacua, or in perturbative expansion.

2.1 Example: a circle

As a simple example of the above ideas, let us consider one dimensional particle on a flat circle[1]. The Hamiltonian is now

$$H = p^2 + V(q) \tag{2.3}$$

where $q \sim q + 4\pi$ (we denote the configuration space $S^1_{4\pi}$). In the simplest case there are two classical vacua. It follows that there are only two possible isometric maps f

$$\begin{aligned} f(q) &= q + 2\pi \pmod{4\pi} \\ f(q) &= 2\pi - q \pmod{4\pi} \end{aligned} \tag{2.4}$$

(this follows from $f(f(q)) = q \pmod{4\pi}$). By arguing about a slightly pushed ground state – a superposition of the ground state and first excited state – we may argue for (2.4).

If we take for granted that the points on the $S^1_{4\pi}$ to be identified are $q \leftrightarrow f(q)$ we may look for an operator Q that maps the state $\psi : S^1_{4\pi} \to \mathbf{C}$ into the another state localized at "same classical points" (but at different q, namely $f(q)$). More precisely, we want that if ψ is a $\hat{q}$-eigenstate, say $\psi(q) = \delta(q - q_0)$, then $Q\psi$ should be nonzero only at $f(q_0)$. Using this "locality" of Q we argue that it is of the form $Q = P(\hat{p})$"σ_x", where "σ_x" $\equiv \exp(i2\pi\hat{p}/\hbar)$ is the translation operator by 2π, and P is a finite polinomial in $\hat{p}$ (so it can only make infinitesimal translations).

[1] In one dimension all metric tensor fields can be made trivial, i.e. $g_{11} = 1$, by appropriate choice of coordinate.

From the requirement that the classical potential $V^{(0)}(q)$ is periodic with period 2π, follows that $W^{(0)}$ must be antiperiodic[2] with period 2π. We now assume the same property for the full W. From that follows

$$\{``\sigma_x", W(\hat{q})\} = 0, \quad (``\sigma_x")^2 = 1 \quad (``\sigma_x")^\dagger = ``\sigma_x" \tag{2.5}$$

Now if we take for Q

$$Q = Q^\dagger = \left(\hat{p} + iW'(\hat{q})\right) ``\sigma_x" \tag{2.6}$$

from (2.1) and (2.5) follows that Hamiltonian (2.3) takes the form

$$H = \frac{1}{2}\{Q, Q\} + E_g = Q^2 + E_g \tag{2.7}$$

If we could find fermion number operator F such that $(-1)^F$ anticommutes with Q, we could say that our starting *purely bosonic* system can be written as supersymmetric. This is our next task.

Locally in q, or perturbatively, we define a fermion number F so that

$$(-1)^F = ``\sigma_z" \tag{2.8}$$

where "σ_z" is defined for the neighbourhoods (of trivial topology) of critical points of $W(q)$ (which are near classical vacua for $\hbar$ small) in the following way. Denote by q_g minimum of $W(q)$ and arrange that $0 < q_g < 2\pi$. Because of the 2π-antiperiodicity of W we know that maximum of W is at $f(q_g) = q_g + 2\pi$. Now, equivalence $q \sim f(q)$ reduces classical configuration space from $S^1_{4\pi}$ to $S^1_{2\pi} = S^1_{4\pi}/\mathbf{Z_2}$. Because quantum corrections break the equivalence, beside "classical position" $\bar{q} \in [0, 2\pi]$ we need another discrete degree of freedom which tells us in which of the classically equivalent points ($\bar{q}$ or $\bar{q} + 2\pi$) particle is. More formaly, we split the wave function $\psi(q)$, $q \in [0, 4\pi\rangle$ in two components $\psi(\bar{q}, \sigma)$, $q \in [0, 2\pi]$, $\sigma = \pm 1$ in the following way

$$\psi(\bar{q}, 1) \equiv \psi(\bar{q}), \quad \psi(\bar{q}, -1) \equiv \psi(\bar{q} + 2\pi), \qquad \bar{q} \in [0, 2\pi] \tag{2.9}$$

From the definition of "σ_x" follows "σ_x"$\psi(q) = \psi(q + 2\pi)$, so we have "σ_x"$\psi(\bar{q}, \sigma) = \psi(\bar{q}, -\sigma)$. We can now define operator "σ_z" such that

$$``\sigma_z"\psi(\bar{q}, \sigma) = \sigma\psi(\bar{q}, \sigma) \tag{2.10}$$

Obvious properties of "σ_z" are

$$\{``\sigma_z", ``\sigma_x"\} = [``\sigma_z", \hat{p}\,] = [``\sigma_z", \bar{q}\,] = 0, \quad (``\sigma_z")^2 = 1, \quad (``\sigma_z")^\dagger = ``\sigma_z"$$

From that, (2.6), and (2.8) trivially follows

$$\{(-1)^F, Q\} = \{``\sigma_z", Q\} = 0$$

[2]We want two *different* classical minima.

Finally, using $\bar{q}$ and "σ_z" instead of q, we can formally write Hamiltonian (2.3) in the standard $N = 2$ SUSY form

$$H = \hat{p}^2 + \big(W'(\hat{\bar{q}})\big)^2 - \hbar W''(\hat{\bar{q}})\text{“}\sigma_z\text{”} + E_g = Q^2 + E_g \tag{2.11}$$

Now, above result is certainly not true and it is easy to find where we cheated. Splitting of configuration space (2.9) imposes specific boundary conditions

$$\psi(2\pi, \sigma) = \psi(0, -\sigma), \quad \psi'(2\pi, \sigma) = \psi'(0, -\sigma)$$

which are obviously incompatible with the definition of "σ_z" (2.10). But, if we restrict ourself to low energy perturbation theory around classical minimum, then boundary conditions became irrelevant and we can consider our *purely bosonic* system to behave as $N = 2$ SUSY theory (2.11).

The same thing can be seen looking at the "smoothness" properties of "σ_z". From its definition we can see that when it acts on eigenvectors of $\hat{q}$, its eigenvalue jumps from 1 to -1 when q passes 2π. From that we can conclude that "σ_z", and so fermion number F also, can be defined only locally around classical minima.

3. Conclusion

Previous analysis would suggest that certain simple assumptions like smoothness of ground state properties in $\hbar$ or vanishing of ground state energy would require supersymmetry. That would mean that it is very difficult to avoid SUSY and if that is necessary because of phenomenological reasons one has to abandon also previously mentioned properties.

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