Non-linear Self-Duality and Supersymmetry

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ABSTRACT: We give a brief review of our recent work on non-linear self-dual $\mathcal{N} = 1, 2$ manifestly supersymmetric gauge theories.

1. Introduction

The original motivation for our work on non-linear self-duality in supersymmetric systems was sparked by some published proposals of the $\mathcal{N} = 2$ supersymmetric Born-Infeld action. Certainly, the requirement of the correct $\mathcal{N} = 0$ limit is a necessary condition, but, as it has already been remarked in the work of Cecotti and Ferrara [1] in the $\mathcal{N} = 1$ case, this is not sufficient. To make the problem well-defined, one should impose additional conditions the supersymmetric Born-Infeld action should satisfy, *e.g.* that it is the effective world-volume action of a D3-brane embedded in six-dimensional Minkowski space-time. [1]

The appropriate approach to arrive at the 'correct' Born-Infeld action would be to look for the action for the Goldstone multiplet of the spontaneous $\mathcal{N} = 4 \rightarrow \mathcal{N} = 2$ breaking. This would be the generalization of the procedure by which Bagger and Galperin [2] (see also [3]) rederived the $\mathcal{N} = 1$ Born-Infeld action of Cecotti and Ferrara, but now with the ambiguity removed. The choice of the Goldstone multiplet is not unique, but if we require that the resulting action allows for the interpretation as the effective world-volume action of a D3 brane in $d = 6$, we have to use the gauge multiplet as the Goldstone multiplet. The reason is that it is the only irreducible $\mathcal{N} = 2$ massless supermultiplet which contains, along with the gauge field, two real scalars and eight real fermionic components, which are interpreted as the Goldstone fields of spontaneaously broken translations transverse to the brane and of the eight spontaneously broken supercharges, respectively.

The generalization of this construction to the $\mathcal{N} = 2$ Born-Infeld action has not been carried out yet, and we have no progress to report on this. The major obstacle is the absence of an off-shell superfield formulation of $\mathcal{N} = 4$ SYM.

An alternative approach, which we will persue, relies on the observation, valid for $\mathcal{N} = 0, 1$, that the Born-Infeld action is invariant under electric-magnetic duality, or, as we will explain, the Born-Infeld action is a solution of the self-duality equation [2]. The self-duality equations were first derived by Gibbons and Rasheed in [4, 5] based on earlier work by Gaillard and Zumino [6, 7, 8] and were generalized to the supersymmetric case, $\mathcal{N} = 1$ and 2, in [9]. Further generalizations with matter fields and/or tensor fields and their supersymmetric extensions, have been reviewed in [10], where references to the original papers can be found.

2. Non-linear electrodynamics

Non-linear electrodynamics is the simplest non-trivial system which exhibits self-duality. It has a long history and goes back to the attempt of

[1] We want to note in passing that a similar ambiguity also exists in the non-abelian generalisation of the Born-Infeld action, supersymmetric or not.

[2] In the literature one often refers to self-dual systems as those systems which are self-dual under Legendre transformations. One can show that solutions to the self-duality equation always possess this property. The approach via the self-duality equations turns out to be much simpler, though.

Born and Infeld to modify Maxwell's equations in such a way that the field and the energy of a point particle stay finite. That led to the Born-Infeld Lagrangian, whose special symmetry properties were first discussed by Schrödinger.

Consider an action $S = \int d^4x\, L$ of the form

$$L(F) = -\frac{1}{4}F^2 + L_{\rm int}(F) \qquad (2.1)$$

and define

$$\tilde{G}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^{\rho\sigma}(F) \equiv 2\frac{\partial}{\partial F^{\mu\nu}}L(F)\,. \qquad (2.2)$$

Since the Bianchi identity $\partial^\mu \tilde{F}_{\mu\nu} = 0$ and the equation of motion $\partial^\mu \tilde{G}_{\mu\nu} = 0$ have the same form, it is natural to consider the *duality transformations*

$$\begin{pmatrix} G'(F') \\ F' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} G(F) \\ F \end{pmatrix} \qquad (2.3)$$

with $AD - BC \neq 0$, $A, B, C, D \in \mathbf{R}$.

There always exists an $L'(F')$ such that $\tilde{G}'(F') = 2\frac{\partial}{\partial F'}L'(F')$. In fact, considering infinitesimal transformations, $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \simeq \mathbf{1} + \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ one finds

$$\begin{aligned} \Delta L &\equiv L'(F) - L(F) \qquad (2.4) \\ &= (a+d)L - \frac{1}{2}d\tilde{G}\cdot F + \frac{1}{4}bF\cdot\tilde{F} - \frac{1}{4}cG\cdot\tilde{G} \end{aligned}$$

where $G\cdot\tilde{G} = G_{\mu\nu}\tilde{G}^{\mu\nu}$, *etc.*

These considerations become non-trivial if one requires *self-duality*, *i.e.* $L'(F) = L(F)$. Note that this does not imply that L itself is duality invariant. In fact, it isn't, but $L - \frac{1}{4}F\cdot\tilde{G}$ is. If one now assumes that *(i)* L is parity even and *(ii)* $L = -\frac{1}{4}F^2 + \mathcal{O}(F^4)$, one finds

- only $U(1) \subset GL(2, \mathbf{R})$ transformations are possible, and
- $L(F)$ needs to satisfy the duality equation

$$F\cdot\tilde{F} + G\cdot\tilde{G} = 0\,. \qquad (2.5)$$

Note that this equation constitutes a strong restriction on $L(F)$.

For the supersymmetric generalizations to which we turn momentarily, it is convenient to rewrite the self-duality equation. For doing this we define

$$\omega = \alpha + i\beta\,,\ \alpha = \frac{1}{4}F\cdot F\,,\ \beta = \frac{1}{4}F\cdot\tilde{F} \qquad (2.6)$$

and

$$L(\omega,\bar\omega) = -\frac{1}{2}(\omega+\bar\omega) + \omega\bar\omega\Lambda(\omega,\bar\omega) = L(\bar\omega,\omega) \qquad (2.7)$$

where the last equality imposes parity invariance and $\Lambda = \text{const} + \mathcal{O}(\omega)$. In terms of Λ the duality equation reads

$$\text{Im}\left\{\frac{\partial(\omega\Lambda)}{\partial\omega} - \bar\omega\left(\frac{\partial(\omega\Lambda)}{\partial\omega}\right)^2\right\} = 0\,. \qquad (2.8)$$

The most prominent (non-trivial) solution of this equation is the Born-Infeld Lagrangian

$$\Lambda_{\rm B.I.} = \frac{g^2}{1 + g^2\text{Re}\omega + \sqrt{1 + 2g^2\text{Re}\omega - g^4(\text{Im}\omega)^2}} \qquad (2.9)$$

which is equivalent to

$$L_{\rm B.I.} = \frac{1}{g^2}\left\{1 - \sqrt{-\det(\eta_{\mu\nu} + gF_{\mu\nu})}\right\} \qquad (2.10)$$

where g is a dimensionful coupling constant. Again with a view to the supersymmetric generalization, we note that $L_{\rm B.I.}$ can be written implicitly as $L_{\rm B.I.} = -\frac{1}{2}(\chi + \bar\chi)$ where the complex field χ satisfies the non-linear constraint $\chi + \frac{1}{2}g^2\chi\bar\chi - \omega = 0$.

For generalizations a) to several $U(1)$ fields, b) couplings to scalars c) and/or (NS,NS) and (R,R) B-fields, d) p-forms in $d > 4$, *c.f.* [10] and references therein.

3. Self-duality and $\mathcal{N} = 1$ SUSY

Again, we will discuss here only the simplest situation: pure non-linear SUSY electrodynamics. Generalizations (*e.g.* coupling to chiral multiplets, (NS,NS) and (R,R) multiplets, tensor multiplets can be found in [10]).

Supersymmetric electrodynamics is described in terms of a chiral (W_α) and an anti-chiral superfield ($\bar{W}_{\dot\alpha}$) [3]. In the following we simply give the $\mathcal{N} = 0 \to \mathcal{N} = 1$ generalization of various necessary quantities. Field strength: $F \to (W, \bar{W})$; action: $S[F] \to S[W, \bar{W}]$; $\tilde{G} = 2\frac{\partial L}{\partial F} \to (iM = 2\frac{\delta S}{\delta W}, -i\bar{M} = 2\frac{\delta S}{\delta \bar{W}})$; Bianchi identity: $dF = 0 \to D^\alpha W_\alpha = \bar{D}_{\dot\alpha}\bar{W}^{\dot\alpha}$, equation of motion:

[3] Our notation is that of [11] and [12], where all necessary details on $\mathcal{N} = 1$ SUSY can be found.

$dG = 0 \rightarrow D^\alpha M_\alpha = \bar{D}_{\dot\alpha} \bar{M}^{\dot\alpha}$. For the duality equation one finds

$$\mathrm{Im} \int d^4x d^2\theta \, (W^2 + M^2) = 0 \, . \qquad (3.1)$$

Any supersymmetric self-dual non-linear electrodynamics must be a solution of eq.(3.1). To solve the self-duality equation we define the anti-chiral superfield $u = \frac{1}{8} D^2 W^2$ and make the ansatz (*c.f.* (2.7))

$$S = \frac{1}{4}\int d^6z W^2 + \frac{1}{4}\int d^6\bar{z} \bar{W}^2 + \frac{1}{4}\int d^8z W^2 \bar{W}^2 \Lambda(u, \bar{u}) \qquad (3.2)$$

In terms of Λ, the self-duality equation can be written in the form

$$\mathrm{Im} \left\{ \frac{\partial(u\Lambda)}{\partial u} - \bar{u} \left(\frac{\partial(u\Lambda)}{\partial u} \right)^2 \right\} = 0 \qquad (3.3)$$

which is to be compared to eq.(2.8). To derive this form of the self-duality equation the use of the Grassmann-oddness of W, i.e. the property $W^3 \equiv 0$, is crucial. If we go to component fields, $W_\alpha \sim \{F_{\mu\nu}, \lambda_\alpha, D\}$, we find $u = \omega - \frac{1}{2} D^2 + \mathcal{O}(\lambda^2)$. We can now use the equations of motion for the photino λ and the auxiliary field D to set them both to zero. This way we get from every solution of eq.(3.1) a solution of eq.(2.5). Turning the argument around, we learn that every self-dual model of the form (2.7) has an extension which is self-dual under manifestly, *i.e.* expressible in terms of superfields, $\mathcal{N} = 1$ supersymmetric duality rotations and which allows for a consistent (with the equations of motion) truncation to the non-supersymmetric model. [4]

Given the non-supersymmetric Born-Infeld action, it is now immediate to find its supersymmetric extension. Defining $A_\pm = \frac{g^2}{8}(D^2 W^2 \pm \bar{D}^2 \bar{W}^2)$ it reads

$$\begin{aligned} S_{\mathrm{B.I.}} &= \frac{1}{4} \int d^6z W^2 + \frac{1}{4} \int d^6\bar{z} \bar{W}^2 \qquad (3.4) \\ &+ \frac{g^2}{4} \int d^8z \frac{W^2 \bar{W}^2}{1 + \frac{1}{2} A_+ + \sqrt{1 + A_+ + \frac{1}{4} A_-^2}} \end{aligned}$$

Similarly as in sect. 2, we can rewrite this as

$$S_{\mathrm{B.I.}} = \frac{1}{4} \int d^6z \, \chi + \frac{1}{4} \int d^6\bar{z} \, \bar{\chi} \qquad (3.5)$$

where the chiral superfield χ is defined via $\chi + \frac{1}{4} \chi \bar{D}^2 \bar{\chi} = W^2$. Even though it is Grassmann even, it is nilpotent, $\chi^2 = 0$, a property which follows from $W^3 = 0$.

We want to stress that while it was quite a burden for Bagger and Galperin to demonstrate self-duality of this action, it is an immediate consequence of our approach. Bagger and Galperin also showed that $S_{\mathrm{B.I.}}$ is the action for the Goldstone multiplet (W) associated with $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ partial supersymmetry breaking. It possesses a second, non-linearly realized supersymmetry. It is, however, not the unique $\mathcal{N} = 1$ extension of the non-supersymmetric Born-Infeld theory: the action is not determined uniquely if one insists on the requirement of the correct $\mathcal{N} = 0$ limit only. But it is the requirement of either self-duality or the presence of a second supersymmetry, which make the action unique.

4. Self-duality and $\mathcal{N} = 2$ SUSY

In this final section, we will discuss self-duality of systems with manifest $\mathcal{N} = 2$ supersymmetry. Again, our main focus will be the Born-Infeld action.

We work in $\mathcal{N} = 2$ superspace with coordinates $\mathcal{Z}^A = (x^a, \theta^\alpha_i, \bar{\theta}^i_{\dot\alpha})$, $i = 1, 2$. The relevant superfield is a chiral field strength $\mathcal{W}$ which is constrained to satisfy the Bianchi identity $\mathcal{D}^{ij} W = \bar{\mathcal{D}}^{ij} \bar{W}$, $\mathcal{D}^{ij} = \mathcal{D}^{\alpha i} \mathcal{D}^j_\alpha$. [5] Given an arbitrary action of the form $S[\mathcal{W}, \bar{\mathcal{W}}]$ we define $i\mathcal{M} \equiv 4 \frac{\delta S}{\delta \mathcal{W}}$ and $-i\bar{\mathcal{M}} \equiv 4 \frac{\delta S}{\delta \bar{\mathcal{W}}}$ in terms of which the equations of motion are $\mathcal{D}^{ij} \mathcal{M} = \bar{\mathcal{D}}^{ij} \bar{\mathcal{M}}$. The $\mathcal{N} = 2$ self-duality equation can be derived in much the same way as before and one finds

$$\int d^8\mathcal{Z} \, (W^2 + M^2) = \int d^8\bar{\mathcal{Z}} \, (\bar{W}^2 + \bar{M}^2) \quad (4.1)$$

Only two solutions to this self-duality equation are known in closed form:

- the $\mathcal{N} = 2$ Maxwell action

$$S = \frac{1}{8} \int d^8\mathcal{Z} \, \mathcal{W}^2 + \frac{1}{8} \int d^8\bar{\mathcal{Z}} \, \bar{\mathcal{W}}^2 \qquad (4.2)$$

- the $\mathcal{N} = 2$ Born-Infeld theory

$$S = \frac{1}{4} \int d^8\mathcal{Z} \, \mathcal{X} + \frac{1}{4} \int d^8\bar{\mathcal{Z}} \, \bar{\mathcal{X}} \qquad (4.3)$$

[4] In the non-linear case there are other, besides the trivial, solutions to the equation of motion for the auxiliary field D. They do, however, not lead to a non-supersymmetric self-dual system [13].

[5] For $\mathcal{N} = 2$ superspace notation we refer to [10] and the references given there.

where the chiral superfield $\mathcal{X}$ satisfies $\mathcal{X} = \mathcal{X}\bar{\mathcal{D}}^4\bar{\mathcal{X}} + \frac{1}{2}\mathcal{W}^2$. This form of the action suffices to demonstrate self-duality; this was done in [10]. A closed form of the solution to the non-linear constraint on $\mathcal{X}$ has not been found yet (of course, there is no problem in constructing the perturbative solution of the constraint). The methods which were successful at $\mathcal{N} = 1$ do not apply, as the property $W^3 \equiv 0$ does not generalize to $\mathcal{W}$. That this systems can be consistently truncated to the $\mathcal{N} = 1$ Born-Infeld action, was demonstrated by Ketov in [14]. In [9] it was shown that, as expected, this requirement does not fix the action uniquely. That it has two non-linearly realized supersymmetries, which would be necessary for this action to qualify as the action for the Goldstone multiplet of partial $\mathcal{N} = 4 \to \mathcal{N} = 2$ SUSY breaking, has not been demonstrated yet. In fact, we now give arguments that this is not so (see also [3, 10]). In fact, the low-energy effective action on the world-volume of D3 branes in $d = 6$ should have precicely this property. Its bosonic piece should have the form

$$L = 1 - \sqrt{-\det(\eta_{\mu\nu} + F_{\mu\nu} + \partial_\mu\phi\partial_\nu\bar{\phi})} \qquad (4.4)$$

where the complex scalar ϕ describes the fluctuations of the brane in its two transverse directions. The action has the expected shift symmetry $\phi \to \phi + \sigma$ for constant complex parameter σ. This identifies ϕ as the Goldstone field associated with broken translation invariance. The $\mathcal{N} = 2$ version of this symmetry has the general form $\mathcal{W}(\mathcal{Z}) \to \mathcal{W}(\mathcal{Z}) + \sigma + \mathcal{O}(\mathcal{W}, \bar{\mathcal{W}})$. The scalar component of the superfield $\mathcal{W}$ is (in genral non-linearly) related to the field ϕ. We have shown explicitly in [10] that the above action, even though it has the correct $\mathcal{N} = 1$ limit, does not have this shift symmetry.

There are two ways to proceed in searching for the correct manifestly supersymmetric world-volume action for the D3 brane in $d = 6$. The first possiblility is the already mentioned generalization of the procedure of Bagger and Galperin of explicitly constructing the action of the Goldstone multiplet with the hidden supersymmetries. An alternative, which was proposed in [14], is to first construct a manifestly (1,0) supersymmetric Born-Infeld action in $d = 6$ and then reduce it to $d = 4$. The shift symmetry would then be built in. But this is not a simple task either, as we indicated in [10].

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