Tools for E_6 model building

Gregory W. Anderson*
Department of Physics and Astronomy, Northwestern University
E-mail: ganderson@northwestern.edu

ABSTRACT: Recent progress in developing tools for E_6 model building is reported. These tools are used to illustrate a natural mechanism for Higgs mixing in E_6 models.

1. Introduction

The exceptional group E_6 has received consideration as a unified group for over twenty years [1, 2, 3]. The fundamental representation allows for the possibility that an entire generation of standard model fermions, a right handed neutrino and two Higgs doublet are unified into a single representation.

$$\begin{aligned}\mathbf{27} = & (\mathbf{3},\mathbf{2})_{\mathbf{1}} \oplus (\bar{\mathbf{3}},\mathbf{1})_{-\mathbf{4}} \\ & \oplus (\mathbf{3},\mathbf{1})_{-\mathbf{2}} \oplus (\bar{\mathbf{3}},\mathbf{1})_{\mathbf{2}} \oplus (\bar{\mathbf{3}},\mathbf{1})_{\mathbf{2}} \\ & \oplus (\mathbf{1},\mathbf{2})_{-\mathbf{3}} \oplus (\mathbf{1},\mathbf{2})_{-\mathbf{3}} \oplus (\mathbf{1},\mathbf{2})_{\mathbf{3}} \\ & \oplus (\mathbf{1},\mathbf{1})_{\mathbf{6}} \oplus (\mathbf{1},\mathbf{1})_{\mathbf{0}} \oplus (\mathbf{1},\mathbf{1})_{\mathbf{0}}\end{aligned} \tag{1.1}$$

Despite its potential, E_6 has been exploited to a much lesser extent than its subgroups $SU(5)$ and $SO(10)$ [4, 5]. In fact, surprisingly little of the group theory necessary to build a complete E_6 model had been explicitly worked prior to the work reported here[6], with the notable exception of the work by Koh, Patera and Rousseau [7].

The principle motivation for the work reported here [6] was to develop a set of tools which are currently being used to construct predictive models of fermion masses and mixing angles in E_6 unified theories [8], but our group theoretic results should be important and useful to anyone seeking to build an explicit unified models based on E_6.

Model Building Considerations

Although a wealth of predictive theories based on smaller groups like $SU(5)$ and $SO(10)$ have been proposed, theories containing at most $SO(10)$ unification are in several respects unsatisfactory. First, from the point of view of supersymmetric theories, the distinction between the standard model fermions and the Higgs doublets as candidates for inclusion in a unified representation is artificial, since they are all contained in left handed chiral superfields. A supersymmetric extension of the one generation standard model would require the chiral superfields listed below:

$$\begin{aligned} SU(5): &\ \mathbf{10} \oplus \overline{\mathbf{5}} \oplus \overline{\mathbf{5}} \oplus \mathbf{5} \oplus \mathbf{1} \\ SO(10): &\ \mathbf{16} \oplus \mathbf{10} \\ E_6: &\ \mathbf{27} \end{aligned} \tag{1.2}$$

¿From this point of view, the unification present in $SU(5)$ and $SO(10)$ is seen to be incomplete.

Second, these theories do not naturally accommodate the fact that

$$\frac{m_d}{m_u}\frac{m_t}{m_b} \gg 1. \tag{1.3}$$

without a loss of predictivity. Moreover, an $SO(10)$ model with a single **10** leads to the GUT scale relation $\lambda_t = \lambda_b = \lambda_\tau$ [9], which implies $\tan\beta \sim m_t/m_b$, which is problematic from the point of view of electroweak symmetry breaking. In order to avoid large $\tan\beta$ one must mix the Higgs doublets in the **10** with other doublet states. In the case where we have n copies of the 10 dimensional representation, the mass matrix for the n doublet pairs H_U and H_D induced at the GUT scale is an $n \times n$ matrix μ:

$$-L \supset H_U \mu H_D. \tag{1.4}$$

In order to have a pair of doublets which do not acquire GUT scale masses we require $\det\mu = 0$.

*Based on work in collaboration with Tomas Blazek

The light H_U (H_D) states are the zero eigenvectors of $\mu^\dagger$ (μ) respectively. The most natural ways to accomplish this are to arrange family symmetries so that a row or column vanish, rows or columns proportional, or other symmetry properties which include, for example, a totally anti-symmetric matrix when $n = 3$. The first and second cases are unnatural and lead to models with little predictivity while mass matrices typical of the latter cases:

$$\mu \sim \begin{pmatrix} & X & \\ \pm X & Z & Y \\ & \pm Y & \end{pmatrix}, \; \mu \sim \begin{pmatrix} & A & B \\ -A & & C \\ -B & -C & \end{pmatrix} \tag{1.5}$$

mix the H_U and H_D sectors identically leading to $\tan\beta \sim m_t/m_b$. As argued below, E_6 provides a natural solution to this problem.

E_6 Clebsch Gordon coefficients

In order to provide a complete construction of a supersymmetric E_6 theory, we have computed all of the Clebsch-Gordon coefficients occuring in in the renormalizable superpotential interactions of an E_6 theory containing fundamental, anti-fundamental and adjoint representations. The relevant renormalizable operators are listed below:

Dimension	Singlets	
2	$\mathbf{27}\cdot\overline{\mathbf{27}}$	$\{\mathbf{78},\mathbf{78}\}$
3	$\{\mathbf{27},\mathbf{27}\}\cdot\mathbf{27}$	$\{\overline{\mathbf{27}},\overline{\mathbf{27}}\}\cdot\overline{\mathbf{27}}$
	$\mathbf{27}\cdot\overline{\mathbf{27}}\cdot\mathbf{78}$	$[\mathbf{78},\mathbf{78}]\,\mathbf{78}$

In addition to the operators listed above, we have determined several of the dimension four operators in our analysis [6]. This calculation was made using standard techniques. The algebra is written as a set of ladder operators:

$$\begin{aligned} [H_a, H_b] &= 0 \\ [H_a, E_{\pm\alpha_i}] &= \pm\alpha_{ia} E_{\pm\alpha_i} \\ [E_{\pm\alpha_i}, E_{\pm\alpha_j}] &= \begin{cases} 0 & \alpha+\beta \text{ not a root} \\ \alpha^a H_a & \alpha = -\beta \\ N_{\alpha\beta} E_{\alpha+\beta} & \text{otherwise} \end{cases} \end{aligned} \tag{1.6}$$

States in a representation are obtained by a succession of lowering operators applied to the highest weight in the representation:

$$(a_1, a_2, \ldots, a_n) \;\xrightarrow{\;E_{-\alpha_j}\;}\; (a_1', a_2', \ldots, a_n'), \qquad a_i' = a_i - A_{ij}$$

The Dynkin coordinates of a weight μ: are $a_i = 2\frac{\langle\mu,\alpha_i\rangle}{\langle\alpha_i,\alpha_i\rangle}$, and the Cartan matrix is:

$$A_{ij} = 2\frac{\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle} \tag{1.7}$$

This procedure proves to be technically challenging for representations with a large number of degenerate weights. The lowering coefficients for degenerate weights are defined by:

$$E_{\pm\alpha_i} \mid \mu_a\rangle = N_{\pm\alpha_i,\mu_a\to(\mu\pm\alpha_i)_b} \mid (\mu\pm\alpha_i)_b\rangle \tag{1.8}$$

These lowering coefficients satisfy the recurrence relation:

$$\begin{aligned} &(G^{(\mu\pm\alpha)})^{-1}_{dc} N_{\pm\alpha,\mu_a\to(\mu\pm\alpha)_c} N^*_{\pm\alpha,\mu_b\to(\mu\pm\alpha)_d} \\ &= \mp(G^\mu)^{-1}_{ba}\langle\alpha,\mu\rangle \\ &+ G^{(\mu\mp\alpha)}_{ef}(G^\mu)^{-1}_{bg}(G^\mu)^{-1}_{ha} N_{\pm\alpha(\mu\mp\alpha)_e\to\mu_g} \\ &\times N^*_{\pm\alpha(\mu\mp\alpha)_f\to\mu_h}, \end{aligned} \tag{1.9}$$

where the metric on the degenerate weight space is

$$\begin{aligned} I_\mu &= G_{ij} \mid \mu_i\rangle\langle\mu_j \mid \\ G_{ij} &= (M^{-1})_{ij} \quad \text{where} \quad M_{ij} = \langle\mu_i \mid \mu_j\rangle. \end{aligned} \tag{1.10}$$

In general, lowering through a chain of degenerate weights involves three metrics:

$$\mu+\alpha \quad G^{(\mu+\alpha)} \;\xrightarrow{\;E_{-\alpha}\;}\; \mu \quad G^{(\mu)} \;\xrightarrow{\;E_{-\alpha}\;}\; \mu-\alpha \quad G^{(\mu-\alpha)}$$

The technical challenge is further aggravated by the fact that it is often impossible to choose a single basis for all of the states of a degenerate weight in a representation such that lowering a particular state results in a particular state of lower weight (as opposed to a linear combination of weights) for all possible lowerings.

The calculation of the product of two adjoint representation provides an example:

$$\mathbf{78} \otimes \mathbf{78} = \mathbf{2430} \oplus \mathbf{2925} \oplus \mathbf{650} \oplus \mathbf{78} \oplus \mathbf{1},$$
$$(000001) \otimes (000001) = (000002) \oplus (001000) \oplus (100010) \oplus (000001) \oplus (000000).$$

level	weight		deg.	
0	(000002)	= (000001)(000001)	1	2430
1	(001000)	$= \{(000001),(00100-1)\}$		
	(001000)	$= \{(000001),(00100-1)\}_\pm$	2	1 2430 1 2925
	⋮			
6	(100010)		8	3 2430 4 2925 1 650
	⋮			
11	(000001)		32	11 2430 15 2925 5 650 1 78
	⋮			
22	(000000)	NB 185 → 36	108	36 2430 45 2925 20 650 6 78 1 1

For the degenerate (000000) weight, the 36 linearly-independent degenerate weights of the **2430** which occur at level 22 must be formed from 185 linearly-dependent degenerate (000000) weights which one obtains after application of the lowering operator to the level 21 states.

Masses and Mixings

The relevant operators for the charged fermion masses and Higgs doublet mixing obtained from our analysis [6], include a totally symmetric dimension three operator:

$$O_{\{abc\}} = \mathbf{27}_a\, \mathbf{27}_b\, \mathbf{27}_c, \tag{1.11}$$

and two dimension-four operators which are respectively symmetric and anti-symmetric with respect to interchange of two fundamental representations

$$\begin{aligned} O_{\{ab\}c} &= \frac{1}{M}\{\mathbf{27}_a, \mathbf{27}_b\}\langle\mathbf{78}\rangle\mathbf{27}_c. \\ O_{[ab]c} &= \frac{1}{M}\left[\mathbf{27}_a, \mathbf{27}_b\right]\langle\mathbf{78}\rangle\mathbf{27}_c, \end{aligned} \tag{1.12}$$

and Frogatt-Nielsen like [10] operators which can be generated by integrating heavy vector-like $\mathbf{27} - \overline{\mathbf{27}}$ pairs out of the theory:

$$O_{abc} = \mathbf{27}_a \frac{\langle\mathbf{78}\rangle_1}{\langle 1,78\rangle_1} \cdots \frac{\langle\mathbf{78}\rangle_n}{\langle\mathbf{1,78}\rangle_n} \mathbf{27}_b \frac{\langle\mathbf{78}\rangle_{n+1}}{\langle\mathbf{1,78}\rangle_{\mathbf{n+1}}} \cdots \frac{\langle\mathbf{78}\rangle_l}{\langle 1,78\rangle_l} \mathbf{27}_c \tag{1.13}$$

These operators, like the analogous operators for $SO(10)$ [11], relate the Yukawa couplings $Y_{u,d,e}$ up to possible Clebsch factors from vevs of the adjoint representation. The larger unification in E_6 gives an additional relation between these couplings, the neutrino Yukawa couplings, and the mixing of the Higgs doublets. Below we list example induced mixings, for $\{\mathbf{27}_i, \mathbf{27}_j\}_\pm \mathbf{27}_k$: induced by the operators above:

$$\begin{aligned} Y_F &= \begin{pmatrix} 0 & C^\pm_{F^c} H_k & \pm C^\pm_{F^c} H_j \\ C^\pm_F H_k & 0 & \pm C^\pm_\phi H_i \\ \pm C^\pm_F H_j & C^\pm_\phi H_i & 0 \end{pmatrix} \\ \mu &= \begin{pmatrix} 0 & C^\pm_H \langle S_k\rangle & \pm C^\pm_H \langle S_j\rangle \\ C^\pm_h \langle S_k\rangle & 0 & \pm C^\pm_S \langle S_i\rangle \\ \pm C^\pm_h \langle S_j\rangle & C^\pm_S \langle S_i\rangle & 0 \end{pmatrix} \end{aligned} \tag{1.14}$$

the anti-symmetric operator is seen to provide a natural solution to $\det\mu = 0$, with unequal mixings for H_U and H_D as advertised.

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