

Wilson loop via AdS/CFT duality*

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ABSTRACT: The Wilson loop in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory admits a dual description as a macroscopic string configuration in the adS/CFT correspondence. We discuss the correction to the quark anti-quark potential arising from the fluctuations of the superstring.

1. Introduction

One important ingredient of the string dualities is the twofold description of D-branes as solitonic supergravity solutions and as submanifolds of spacetime where open strings may end. The second description leads to a gauge field theory on the world-volume of D-branes. Based on this general idea is Maldacena's conjecture[1] relating superconformal field theories to supergravity (or superstrings) living in a higher dimensional space with boundary. (For a review containing also a comprehensive list of references, see [2].) For example, $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang Mills theory in four dimensions is dual to the type IIB string theory on $adS_5 \times S^5$ space. The radii of the S^5 and the adS_5 are equal and related to the Yang-Mills coupling via $R^2/\alpha' = \sqrt{4\pi g_{YM}^2 N}$. (The string coupling g is $g = g_{YM}^2$.) For supergravity to be a good effective description, we need the radius of curvature to be large and also the string coupling to be small. This means that we need g_s small but $g_s N$ large. The latter is however the 't Hooft coupling constant in the large N field theory which is thus strongly coupled. Maldacena's conjecture provides a new possibility to gain insight into strongly coupled gauge theory by studying weakly coupled string theory. As an application,

Wilson loops have been computed in [3] and [4]. The string configuration for a quark-antiquark pair separated by a distance L , is a long string on $adS_5 \times S^5$, the ends of which are restricted to the (four dimensional) boundary of adS_5 , where they are at a distance L apart. The expectation value of the Wilson loop is then given by the effective energy of the string. We will review this computation in the next section. In the third section leading corrections are discussed. As a further application of our techniques we briefly review in section four membrany corrections to the Wilson surface in M5 theory. We conclude with a short summary.

2. Review

The dual description of $\mathcal{N} = 4$ super Yang-Mills theory is a type IIB string living in $adS_5 \times S^5$. In particular the target space metric (G_{MN}) is

$$ds^2 = R^2 \left[U^2 \left(-(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right) + \frac{dU^2}{U^2} + d\Omega_5^2 \right]. \quad (2.1)$$

Compared to [4] we have rescaled $U \rightarrow R^2 U$, and also set $\alpha' = 1$. In addition there is a constant dilaton and N units of RR-4-form flux through S^5 . In order to compute the Wilson loop one

*Talk given by S. Förste.

minimizes the Nambu-Goto action [4, 3]¹

$$S_{NG} = \int \frac{d^2\sigma}{2\pi} \sqrt{-\det G_{MN} \partial_i X^M \partial_j X^N} \quad (2.2)$$

with the boundary condition that the ends of the string are separated by a distance L at $U = \infty$. We work in the static gauge ($X^0 = \tau$, $X^1 = \sigma$) and restrict to the case that all coordinates but U are constant. The radial coordinate U depends on σ . Then the (implicit) solution reads²

$$\partial_\sigma U = \pm \frac{U^2}{U_0^2} \sqrt{U^4 - U_0^4}, \quad (2.3)$$

where

$$U_0 = \frac{(2\pi)^{\frac{3}{2}}}{\Gamma(\frac{1}{4})^2} L \quad (2.4)$$

is the minimal U -value the string obtains. The energy of the quark-antiquark pair is the length of the geodesic (open string) connecting them. One finds (after subtracting an L independent infinite contribution from the self energy of the heavy quarks)

$$E = -\frac{4\pi \sqrt{g_{YM}^2 N}}{\Gamma(1/4)^4} L. \quad (2.5)$$

This strong coupling result differs from the perturbative field theory computation ($g_{YM}^2 N$ small), which predicts a Coulomb law with $E \sim g_{YM}^2 N/L$. In general the numerator is a function of $g_{YM}^2 N$ which interpolates between these two forms, and hence there ought to be corrections to (2.5) which is the result of a classical supergravity computation. Since $adS_5 \times S^5$ is an exact string background [5, 6, 7], the first correction comes from the fluctuations of the superstring (R^2/α' correction). In this talk we will report on work in this direction[8]. Corrections due to string fluctuations have been discussed in [9], and subsequently in [10, 11, 12, 13, 14]. Ref.[15] considered corrections to the field theoretical result.

3. Fluctuations

The quantum theory of type IIB string in this background is described by the action in [6]. It

¹After switching off the world-sheet fermions, the type IIB action reduces to the Nambu-Goto action.

²In the following we will just take the upper sign with the understanding that the square root stands for both branches.

is a Green-Schwarz type sigma model action on a target supercoset. The usual sigma model expansion in R^2/α' results in a power series in $1/\sqrt{Ng_{YM}^2}$. Since conformal invariance prevents the appearance of a new scale these corrections are not expected to change the $1/L$ dependence of E on dimensional grounds. However they can modify the ‘Coulomb charge’.

The leading order correction is obtained by expanding around the classical configuration (2.3) to second order in fluctuations. We parameterize the bosonic fluctuations by normal coordinates[16]: ξ^a on $adS_5 \times S^5$, (here $a = 0, \dots, 4; 5, \dots, 9$ are local (flat) Lorentz indices; ξ^4 is in the U direction). Using the normal coordinate expansion one ensures that the functional measure for the fluctuations is translation invariant. This takes care of potential problems due to the curved target space. Ref. [14] has an extensive discussion on additional subtleties in the functional measures due to a curved world sheet.

At second order, the bosonic fluctuations in adS_5 and S^5 directions and the fermionic fluctuations decouple. Before writing their action, we define the combinations

$$\begin{aligned} \xi^\parallel &= \frac{U_0^2}{U^2} \xi^1 + \frac{\sqrt{U^4 - U_0^4}}{U^2} \xi^4, \\ \xi^\perp &= -\frac{\sqrt{U^4 - U_0^4}}{U^2} \xi^1 + \frac{U_0^2}{U^2} \xi^4, \end{aligned} \quad (3.1)$$

which parameterize fluctuations along the longitudinal and perpendicular directions of the background string in the one-four plane. Now the adS_5 part of the action becomes

$$\begin{aligned} S_{adS}^{(2)} &= \frac{1}{4\pi} \int d^2\sigma \sqrt{-h} \left[h^{ij} \left(\sum_{a=2,3,\perp} \partial_i \xi^a \partial_j \xi^a \right) \right. \\ &\quad \left. + 2(\xi^2)^2 + 2(\xi^3)^2 + 2 \frac{U^4 - U_0^4}{U^4} (\xi^\perp)^2 \right] \end{aligned} \quad (3.2)$$

where h_{ij} is (up to a factor of R^2) the classical induced world-sheet metric³

$$ds^2 = -U^2 (d\sigma^0)^2 + \frac{U^6}{U_0^4} (d\sigma^1)^2. \quad (3.3)$$

³For our purpose it is more convenient to work with the induced metric rather than the standard (conformally) flat one on the world-sheet.

Observe that ξ^0 and ξ^\parallel do not appear in (3.2) (total derivatives have been dropped). Hence a natural choice to fix world-sheet diffeomorphisms is

$$\xi^0 = \xi^\parallel = 0. \quad (3.4)$$

The action quadratic in fluctuations in S^5 directions comes out to be

$$S_{S^5}^{(2)} = \frac{1}{4\pi} \int d^2\sigma \sqrt{-h} h^{ij} \sum_{a=5}^9 \partial_i \xi^a \partial_j \xi^a. \quad (3.5)$$

The fermionic part of the action is given by plugging in the background (2.4) into the action of [6] and keeping terms quadratic in fermions. This action has local fermionic κ -symmetry which has to be fixed for the one-loop calculation. There is a proposal in the literature[17] to this end. For our purpose, however, it turns out that the following choice is most convenient.⁴ We will set (in the notation of [6])

$$(\gamma^-)^\alpha_\beta \theta^{1\beta\beta'} = 0 \quad , \quad (\gamma^+)^\alpha_\beta \theta^{2\beta\beta'} = 0 \quad (3.6)$$

where $\gamma^\pm = \gamma^0 \pm \gamma^\parallel$ with $\gamma^\parallel = \frac{U_0^2}{U^2} \gamma^1 + \frac{\sqrt{U^4 - U_0^4}}{U^2} \gamma^4$ (Cf. (3.1)). With this choice the target space spinors ‘metamorphose’ into world-sheet spinors, and the action is found to simplify substantially. The corresponding equations of motion are most compactly written for the ‘two-component’ world-sheet spinors $\begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix}$:

$$\left[i (\rho^m \nabla_m)^\alpha_\beta - \delta^\alpha_\beta \rho^3 \right] \begin{pmatrix} \theta^{1\beta\alpha'} \\ \theta^{2\beta\alpha'} \end{pmatrix} = 0. \quad (3.7)$$

The notation needs explanation. Firstly, the covariant derivatives act as

$$(\nabla_\pm \theta^I)^{\alpha\alpha'} = \left[\delta^\alpha_\beta \left(\partial_\pm \pm \frac{\omega_\pm}{2} \right) + (A_\pm)^\alpha_\beta \right] \theta^{I\beta\alpha'}, \quad (3.8)$$

where the tangent space derivatives

$$\partial_\pm = \frac{1}{U} \partial_\tau \pm \frac{U_0^2}{U^3} \partial_\sigma \quad (3.9)$$

are defined with the help of a (inverse) zweibein ϵ_m of the metric (3.3), $\omega_m^{01} = \epsilon_m^\tau \omega_\tau^{01}$ being the

⁴We will comment on a different choice below.

corresponding spin connection. There is an additional gauge field

$$A_\pm = \pm \frac{U_0^2}{U^2} \gamma^{14} \quad (3.10)$$

for local rotations in the tangent one-four-plane. Finally, the matrices

$$\rho^+ = \begin{pmatrix} 0 & 0 \\ \gamma^0 & 0 \end{pmatrix} \quad , \quad \rho^- = \begin{pmatrix} 0 & \gamma^0 \\ 0 & 0 \end{pmatrix} \quad (3.11)$$

satisfy a two dimensional Clifford algebra, and $\rho^3 = [\rho^+, \rho^-]$. The fermionic action is easily inferred from the equations of motion (3.7).

Collecting our results (3.2), (3.5) and (3.7) one can write a formal expression for the 1-loop contribution to the effective action as a ratio of determinants of two dimensional generalized Laplace operators[8]. These determinants suffer from divergences and can be regularized by, say, the heat kernel technique[18]. The quadratic divergences cancel, but naively a logarithmic divergence remains, which may be absorbed in the (infinite) mass of the external quarks[8]. As unsatisfactory as it may be, it does not affect our result for the correction to the Coulomb charge. More recently the authors of [14] have argued that this divergence should, as in flat space, actually cancel. As far as the corrections to the Coulomb charge are concerned the results of [14] and ours [8] are actually equivalent. In the following we demonstrate that the apparently different expressions for the fermionic operators in [8] and [14] are related to each other by a local Lorentz rotation⁵. To this end, define

$$\theta^I = S \psi^I, \quad (3.12)$$

where we suppressed target space spinor indices. For the matrix S we choose the one given in Ref.[14], Section 6.3,

$$S = \cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \gamma^{14}, \quad (3.13)$$

where

$$\cos \alpha = \frac{U_0^2}{U^2}, \quad (3.14)$$

$$\sin \alpha = \frac{\sqrt{U^4 - U_0^4}}{U^2}, \quad (3.15)$$

⁵From the sigma model point of view this is just a field redefinition with unit Jacobian.

implying that

$$\partial_\sigma \alpha = 2U. \quad (3.16)$$

The Dirac operator for ψ is given by conjugation by S from the one for θ . Using the fact that S commutes with the ρ_m and using ((3.9), (3.10))

$$S^{-1}A_\pm S = -S^{-1}\partial_\pm S = \pm \frac{U_0^2}{U^2}\gamma^{14}, \quad (3.17)$$

we find that the Dirac operator acting on the fields ψ^I (3.12) takes again the form (3.7), but now the connection A_\pm has been gauged away. It is also easy to show that for the redefined spinors ψ^I the kappa-fixing condition takes the form (3.6), but with

$$\gamma^\parallel \rightarrow S^{-1}\gamma^\parallel S = \gamma^1, \quad (3.18)$$

which is the same as (6.35) in [14]. This shows that the results in [14] are equivalent to ours.

4. Wilson surface in M5 theory

The Maldacena conjecture also applies to the case of M-theory living on $adS_7 \times S^4$ being dual to the field theory on a stack of M5 branes[1]. The metric of $adS_7 \times S^4$ is

$$ds^2 = l_p^2 R^2 \left[U^2 \eta_{\mu\nu} dx^\mu dx^\nu + 4 \frac{dU^2}{U^2} + d\Omega_4^2 \right], \quad (4.1)$$

where we have rescaled the five-brane coordinates x^μ by $R^{3/2}$ as compared to[1], $d\Omega_4^2$ is the metric on S^4 . In the limit that the eleven dimensional Planck length l_p goes to zero M-theory on $adS_7 \times S^4$ is conjectured to be dual to the field theory on N M5-branes, where the adS radius and the number of five-branes are related,

$$R = (\pi N)^{\frac{1}{3}}. \quad (4.2)$$

Higher curvature corrections will be small as long as N is taken to be large. In analogy to the previously discussed Wilson loop one can study a Wilson surface in M5-theory. The set up is a membrane extending along the $x^{0,1,2}$ and the U direction ending in two parallel lines separated by a distance L at the boundary of adS_7 [4]. In the following we will recall this set-up (with slightly changed conventions) and thereafter study corrections due to fluctuations of the membrane.

This will be a brief summary of the work presented in[19]. The classical background corresponding to the Wilson surface is obtained by minimizing the world volume of the membrane

$$S = \frac{1}{2\pi} \int \sqrt{-\det(G_{MN}\partial_a X^M \partial_b X^N)} \quad (4.3)$$

with appropriate boundary conditions. The indices M, N label the eleven target space coordinates and a, b are world volume coordinates (τ, σ, ϕ) . By choosing the static gauge $X^0 = \tau$, $x^1 = \sigma$, $X^2 = \phi$ and assuming all other coordinates but $U = U(\sigma)$ to be constant one finds the solution,

$$\partial_\sigma U = \pm \frac{U^2}{2U_0^3} \sqrt{U^6 - U_0^6}. \quad (4.4)$$

Requiring that the membrane ends in two parallel strings at distance L determines the integration constant

$$U_0 = \frac{2}{3L} B\left(\frac{2}{3}, \frac{1}{2}\right), \quad (4.5)$$

where B denotes Euler's beta-function. Computing the vacuum energy density of this configuration one obtains (again after subtracting an L independent infinite contribution to the self-energy of the strings) the potential between the two strings living on the M5-branes,

$$\varepsilon_{pot} = -\frac{2R^3}{27\pi} B\left(\frac{2}{3}, \frac{1}{2}\right)^3 \frac{1}{L^2}. \quad (4.6)$$

This result is reliable for large N where the geometry is not corrected and the classical approximation dominates. In [20] it was argued that there are no corrections to the geometry due to finite N . Another potential source for corrections are fluctuations of the membrane around its classical background described above. Again we expand in normal coordinates[16] and obtain the action quadratic in bosonic fluctuations. We trade the fluctuations in one- and six-direction⁶ for normal and parallel ones,

$$\xi^\parallel = \frac{U_0^3}{U^3} \xi^1 + \frac{\sqrt{U^6 - U_0^6}}{U^3} \xi^6 \quad (4.7)$$

$$\xi^\perp = -\frac{\sqrt{U^6 - U_0^6}}{U^3} \xi^1 + \frac{U_0^3}{U^3} \xi^6. \quad (4.8)$$

⁶The fluctuations are labeled by Lorentz indices; ξ^6 is in the U direction.

The contribution from the adS directions is

$$S_{adS}^{(2)} = \frac{1}{4\pi} \int d^3\sigma \sqrt{-h} \left[h^{ij} \sum_{a=3}^{5,\perp} \partial_i \xi^a \partial_j \xi^a + \frac{3}{4} \sum_{a=3}^5 (\xi^a)^2 + \left(\frac{9}{4} - R^{(3)} \right) (\xi^\perp)^2 \right] \quad (4.9)$$

where h_{ij} is (up to a factor of R^2) the induced metric,

$$ds^2 = -U^2 d\tau^2 + \frac{U^8}{U_0^6} d\sigma^2 + U^2 d\phi^2, \quad (4.10)$$

and $R^{(3)}$ is the corresponding scalar curvature,

$$R^{(3)} = \frac{3U^6 + U_0^6}{2U^6}. \quad (4.11)$$

Again the three longitudinal directions 0, 2, \parallel drop out of the action and we gauge them to zero. The bosonic fluctuations in S^4 direction have a simple action,

$$S_S^{(2)} = \frac{1}{4\pi} \int d^3\sigma \sqrt{-h} h^{ij} \sum_{a=7}^{10} \partial_i \xi^a \partial_j \xi^a. \quad (4.12)$$

In order to discuss the fermionic fluctuations we take the κ symmetric action of [21]. For our background the part quadratic in fermions consists out of terms containing Γ^a ($a = 0, \dots, 6$), where Γ^a is an eleven dimensional gamma matrix. These can be written as $\Gamma^a = \gamma^a \otimes \gamma^{5'}$ where the lower case gammas are gamma matrices in the tangent spaces of adS_7 and S^4 , respectively. We split the 32-component spinors into two 16-component spinors (θ^1, θ^2) according to their eigenvalue with respect to $\gamma^{5'}$. ($\gamma^{5'} \theta^I = -(-)^I \theta^I$.) A convenient kappa-fixing condition turns out be

$$\left(1 - (-1)^I \gamma^{0\parallel 2} \right) \theta^I = 0, \quad (4.13)$$

where now $\gamma^\parallel = \frac{U_0^3}{U^3} \gamma^1 + \frac{\sqrt{U^6 - U_0^6}}{U^3} \gamma^6$. Imposing the kappa-fixing condition we find that the equations of motion for e.g. θ^1 are

$$\rho^a e_a^i \left(\partial_i + \frac{1}{4} \omega_i^{bc} \rho_{bc} + A_i \right) \theta^1 = -\frac{3}{4} \theta^1, \quad (4.14)$$

where e_a^i and ω_i^{bc} are the vielbeine and spin-connections computed from (4.10) (for details see [19]). The matrices ρ are

$$\rho^0 = \gamma^0, \rho^1 = \gamma^{02}, \rho^2 = \gamma^2 \quad (4.15)$$

satisfying a 3d Clifford algebra. The field $A_\sigma = \frac{3U}{4} \gamma^{16}$ is a background value for a gauge connection belonging to local rotations in the 1-6 plane. (For θ^2 one obtains the same result but with a minus sign in the definition of ρ^1 .) Note that the Dirac operator appearing in (4.14) is covariant from a world volume perspective. Collecting the results (4.9), (4.12) and (4.14) one can express the contribution to the energy density in terms of determinants of covariant operators. These can be analyzed using for example the heat-kernel method [18]. In difference to the previously discussed string case one finds divergent contributions as well to the self-energy density as to the potential energy density [19]. It would be interesting to investigate whether one can extend the arguments of [14] to cancel those divergences. (Since the discussion in [14] is quite heavily based on conformal invariance and 2d calculus this is not straightforward.) Finally, let us point out that also in the membrane case one can gauge away the connection appearing in (4.14). This can be achieved by a field redefinition $\theta^I = S\psi^I$ with

$$S = \cos \frac{\alpha}{2} - \sin \frac{\alpha}{2} \gamma^{16}, \quad (4.16)$$

where

$$\begin{aligned} \cos \alpha &= \frac{U_0^3}{U^3}, \\ \sin \alpha &= \frac{\sqrt{U^6 - U_0^6}}{U^3}. \end{aligned} \quad (4.17)$$

The kappa-fixing condition is again (4.13) but with γ^\parallel replaced by γ^1 . This coincides with the kappa-fixing condition advertised in [22].

5. Summary

In this talk we presented techniques for computing stringy corrections to the Wilson loop in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. The final result can be expressed in terms of determinants of 2d covariant operators. A result for the leading correction to the Coulomb charge in terms of a number is still missing (a rough estimate can be found in [14]). We commented on the relation between our results and those obtained in [14]. In the end we reviewed the application of our techniques to the case of a Wilson

surface in M5-theory. There the result is less satisfactory as divergences also affect the Coulomb charge.

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