Quantum and Braided Integrals

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ABSTRACT: We give a pedagogical introduction to integration techniques appropriate for non-commutative spaces. A rather detailed discussion outlines the motivation for adopting the Hopf algebra language. We then present some trace formulas for the integral on Hopf algebras and show how to treat the $\int 1 = 0$ case. We extend the discussion to braided Hopf algebras relying on diagrammatic techniques. The use of the general formulas is illustrated by explicitly worked out examples.

KEYWORDS: quantum integration, braided integration, non-commutative geometry.

1. Introduction

It is a recurrent theme of recent years that the physics of the not-so-distant future will probably involve some species of 'quantum space', a fuzzy substratum free of the singularities inherent in the classical concept of a point. Then, if a long tradition continues, the algebraic codification of its properties will entail non-commutativity as an essential ingredient. The first textbooks on this new physics will probably include an appendix with elements of integration techniques on the relevant non-commutative spaces. We know little yet about what these spaces might turn out to be but there are general arguments that the coordinate functions over them generate Hopf algebras. While contemplating on the rest of the contents of the book, we attempt here a first sketch of a part of its appendix—a short, informal crash course in some techniques in quantum integration.

Sec. 2 motivates the algebraic setting chosen. Hopf algebras are close enough to classical groups to guarantee a continuity of language and yet accommodate naturally rather exotic geometrics. Assuming a convinced reader, Sec. 3 then goes on with the basics of Hopf algebra integration, while Sec. 4 and 5 extend the discussion to the braided case. We rely on simple, detailed examples to illustrate the proposed techniques.

2. Quantum Points + Translations = Hopf Algebras

2.1 Quantum points

Once we have decided to abandon classical manifolds we have to make up our mind on what kind of spaces we would like to consider. It is common place by now to emphasize that, having admitted non-commutativity in the algebra of functions over the 'space' under consideration, we can no longer talk about an underlying manifold consisting of points—what we are left with is the algebra of functions itself. What needs perhaps to be also stressed, to dispel a certain feel of unease that comes with a pointless space, is that, as we will see in the case of Hopf algebras, the 'manifold' still consists of well-defined entities, which one could think of as 'quantum points'. To make this more precise, consider a non-commutative, in general, algebra $(\mathcal{A}, m, 1_{\mathcal{A}})$, where $\mathcal{A}$ is a vector space and m a multiplication map, which will play in the sequel the role of the algebra of functions over some 'quantum manifold'—we will denote its elements by a, b, *etc.*; $1_{\mathcal{A}}$ is the unit function. We concetrate on the conceptual aspects of the problem and take $\mathcal{A}$ to be finite-dimensional, avoiding potential divergences of the sums that enter in our discussion. Nevertheless, we find the picture of a continuous group manifold invaluable in developing some intuition and, although the latter involves an infinite-dimensional algebra of functions, we will keep it in the back of our mind as a guide. Dual to $\mathcal{A}$, in a sense to be made precise shortly, is a 'quantum space' $\mathcal{H}$, with elements g, h, x, y *etc.*—these are the 'quantum points' referred to earlier and among them there is an identity $1_{\mathcal{H}}$. Functions evaluated on points give numbers, *even in the quantum case.* We formalize this by introducing an *inner product*, or *pairing*, $\langle\cdot,\cdot\rangle$ between $\mathcal{A}$ and $\mathcal{H}$, with values in $\mathbb{C}$, via $\langle h, a\rangle \equiv a(h) \in \mathbb{C}$ for any function $a \in \mathcal{A}$ and 'point' $h \in \mathcal{H}$.

One might ask at this stage, how can functions valued in $\mathbb{C}$ be non-commutative? The answer lies actually in the dual. To see this, consider a classical point h_{cl} and evaluate on it the product of functions ab: $(ab)(h_{cl}) = a(h_{cl})b(h_{cl})$ which is equal to $(ba)(h_{cl})$. What happens is that when a classical point sees a product of functions, it 'splits' in two copies of itself, $h_{cl} \to h_{cl} \otimes h_{cl}$, and feeds each of them as argument to each of the factors in the product

$$
\begin{aligned}
(ab)(h_{cl}) &\equiv (a \otimes b)(h_{cl} \otimes h_{cl}) \\
&= a(h_{cl})b(h_{cl})\,. \qquad (2.1)
\end{aligned}
$$

If the underlying manifold consists in its entirety of such classical points, the functions ab and ba agree when evaluated on all points and can (and should) therefore be considered equal. The conclusion is that the function algebra over a classical space is commutative because the classical *coproduct* map

$$
\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}; \qquad h_{cl} \mapsto h_{cl} \otimes h_{cl} \qquad (2.2)
$$

is symmetric under the exchange of its two tensor factors. To put our notation in some use, we rewrite (2.1)

$$
\begin{aligned}
\langle h_{cl}, ab\rangle &\equiv \langle h_{cl}, m(a \otimes b)\rangle \\
&= \langle \Delta(h_{cl}), a \otimes b\rangle \\
&= \langle h_{cl} \otimes h_{cl}, a \otimes b\rangle \\
&= \langle h_{cl}, a\rangle \langle h_{cl}, b\rangle \\
&= a(h_{cl})b(h_{cl})\,. \qquad (2.3)
\end{aligned}
$$

Notice how, in the second line above, the coproduct map Δ in $\mathcal{H}$ is dual to the product map m in $\mathcal{A}$. We see that, in some sense, classical points are quite primitive, in that the only information they carry is about their own position—when confronted with products of functions they can only produce multiple copies of themselves. Quantum points can do better than this. When paired with products of functions they split, via a coproduct map as above, in two other quantum points, $h \mapsto \Delta(h) \equiv h_{(1)} \otimes h_{(2)}$, with $h_{(1)} \otimes h_{(2)} \neq h_{(2)} \otimes h_{(1)}$ in general, and feed each of them in the two factors of the product

$$
\begin{aligned}
\langle h, ab\rangle &= \langle h_{(1)}, a\rangle \langle h_{(2)}, b\rangle \\
&\neq \langle h_{(2)}, a\rangle \langle h_{(1)}, b\rangle \\
&= \langle h, ba\rangle\,. \qquad (2.4)
\end{aligned}
$$

Consequently, $ab \neq ba$ in general. By examining all the values of all the functions in $\mathcal{A}$, one can nevertheless establish commutation relations between them, *e.g.* if ba systematically returns twice the value of ab, on all quantum points, one imposes the relation $ba = 2ab$ in the algebra. The rule for assigning the two points $h_{(1)} \otimes h_{(2)}$ to h (*i.e.* the coproduct $\Delta(h)$) cannot of course be arbitrary. If the product in $\mathcal{A}$ is to be associative, Δ has to be *coassosiative*

$$
(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta\,. \qquad (2.5)
$$

The identity point is taken to be classical (or *grouplike*): $\Delta(1_{\mathcal{H}}) = 1_{\mathcal{H}} \otimes 1_{\mathcal{H}}$. We introduce also a *counit* ϵ in $\mathcal{H}$. $\epsilon(h)$ is defined to be the value of the unit function on h: $\epsilon(h) \equiv \langle h, 1_{\mathcal{A}}\rangle$. This can be different from 1 since h can be an arbitrary linear combination of elements in $\mathcal{H}$. Just like $(\mathcal{A}, m, 1_{\mathcal{A}})$ defines an algebra, the triple $(\mathcal{H}, \Delta, \epsilon)$ defines a *coalgebra.* Notice that we don't have, at this point, any notion of product of quantum points—$\mathcal{H}$ is not yet an algebra (and, similarly, $\mathcal{A}$ is not yet a coalgebra).

Experimenting a little with the above, one discovers that $\Delta(h)$ has, in general, to involve a sum over pairs of points, rather than a single pair. We continue to denote such a sum by $h_{(1)} \otimes h_{(2)}$ *i.e.*

$$\Delta(h) = \sum_i h^i_{(1)} \otimes h^i_{(2)} \equiv h_{(1)} \otimes h_{(2)} \,. \qquad (2.6)$$

The coassociativity mentioned above says that

$$\begin{aligned} h_{(1)(1)} \otimes h_{(1)(2)} \otimes h_{(2)} &= h_{(1)} \otimes h_{(2)(1)} \otimes h_{(2)(2)} \\ &\equiv h_{(1)} \otimes h_{(2)} \otimes h_{(3)} \,, \end{aligned} \qquad (2.7)$$

where in the last line we have renumbered sequentially the subscripts, given that the particular order of applying the successive Δ's does not matter. There are two points that should be kept in mind to avoid confusion with the above notation: first, $h_{(1)} \otimes h_{(2)}$ in (2.6) and in the second line of (2.7) denotes two different things and second, there are $n-1$ implied summations in $h_{(1)} \otimes \ldots \otimes h_{(n)}$.

2.2 Translations

We continue building up an arena for physics by introducing a concept of translations for our functions—it is in terms of this that the invariance of our integral will be expressed (there are of course other uses for it as well). Quite generally, one can describe (left) translated functions by introducing a new algebra $\mathcal{T}$ that acts on the points in $\mathcal{H}$ via a *left action*

$$\begin{aligned} \triangleright : \quad \mathcal{T} \otimes \mathcal{H} &\to \mathcal{H} \,, \\ t \otimes h &\mapsto t \triangleright h \,, \end{aligned} \qquad (2.8)$$

with

$$(tu) \triangleright h = t \triangleright (u \triangleright h) \,, \qquad (2.9)$$

$t, u \in \mathcal{T}$ and $h \in \mathcal{H}$. Then the translated function $a \in \mathcal{A}$ by t, which we write as a_t, is defined via $a_t(h) \equiv a(t \triangleright h)$. The simplest choice for $\mathcal{T}$ is $\mathcal{H}$ itself with $\triangleright$ a left multiplication, in other words we can endow $\mathcal{H}$ with an associative product (turning it into an algebra) and define $a_h(g) \equiv a(hg)$. From this we can abstract the notion of an 'indefinitely translated' function $a_{(\cdot)}(\cdot)$, where the argument in the subscript defines the translation and the second argument evaluates the translated function. Such a function can be written as a sum over tensor products of single-argument functions

$$\begin{aligned} a_{(\cdot)}(\cdot) &\equiv \sum_i a^i_{(1)}(\cdot) \otimes a^i_{(2)}(\cdot) \\ &\equiv a_{(1)} \otimes a_{(2)} \,, \end{aligned} \qquad (2.10)$$

which we realize as a coproduct in $\mathcal{A}$. Coassociativity of this coproduct is dual to the associativity of the product in $\mathcal{H}$, which itself guarrantees the general property (2.9) of an action. Δ can equally well be thought of as describing translations from the right, in this case the element of $\mathcal{H}$ that describes the translation is fed in the second tensor factor and the resulting right-translated function accepts arguments in the first. We can also introduce a counit ϵ in $\mathcal{A}$: $\epsilon(a)$ is the value of a at the identity. Since the identity point is classical, ϵ supplies a one-dimensional representation for $\mathcal{A}$: $\epsilon(ab) = \epsilon(a)\epsilon(b)$. Notice that, again, product and coproduct are dual

$$\langle hg, a \rangle = \left\langle h \otimes g, a_{(1)} \otimes a_{(2)} \right\rangle \,. \qquad (2.11)$$

It is rather natural to impose on translations a certain covariance property: they should respect the 'quantum nature' of $\mathcal{A}$ *i.e.* the 'indefinitely' translated functions should obey the same commutation relations as the original ones. This implies that the coproduct should be an algebra homomorphism

$$\begin{aligned} \Delta(ab) &= \Delta(a)\Delta(b) \\ (ab)_{(1)} \otimes (ab)_{(2)} &= a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)} \,. \end{aligned} \qquad (2.12)$$

By looking at $\langle hg, ab \rangle$ one finds that the same should hold in $\mathcal{H}$—this turns $\mathcal{A}$ and $\mathcal{H}$ into *bialgebras*. What is missing in order to turn both of them into full-fledged *Hopf algebras* is the *antipode* S. This can be useful if, after translating our functions by some h, we change our mind and wish to undo the translation. When $\mathcal{H}$ is a classical group (discrete or not), $S(h) = h^{-1}$ and $\mathcal{A}$ inherits an antipode via $S(a)(h) \equiv a(S(h))$. In the general case we define S via

$$S(h_{(1)})h_{(2)} = \epsilon(h) = h_{(1)}S(h_{(2)}) \qquad (2.13)$$

(similarly for $\mathcal{A}$) and it still holds

$$\langle S(h), a \rangle = \langle h, S(a) \rangle \,. \qquad (2.14)$$

S is an antihomomorphism, $S(hg) = S(g)S(h)$. Notice that S^2 is not necessarily the identity map; this can be a nuissance in transcribing classical results that hold for groups in the general Hopf algebra case but it is also the source of unexpected novelties. We will often pick a *linear* basis $\{f^i\}$, $i = 0, \ldots, N$ in $\mathcal{A}$. This means that any function in $\mathcal{A}$ can be written as a linear combination, with coefficients in $\mathbb{C}$, of the f^i—we choose $f^0 = 1_{\mathcal{A}}$, the unit function. Similarly, $\{e_i\}$, $i = 0, \ldots, N$, will be a dual linear basis in $\mathcal{H}$ with $\langle e_i, f^j \rangle = \delta_i{}^j$; e_0 will denote the identity $1_{\mathcal{H}}$.

To summarize, we started from a primitive notion of 'quantum space' and the functions defined over it and saw that, in general, the former forms a coalgebra while the latter an algebra. Further introducing translations and requiring their compatibility with the algebra of functions results in a symmetric structure, turning both the space and the functions over it into bialgebras. Adding a notion of inverse we end up with a pair of dual Hopf algebras. We illustrate the above concepts in the following two examples.

Example 2.1 *Discrete classical space*

$\mathcal{H}$ in this case is a discrete group algebra. It contains n classical points $\{e_i\}$, $i = 0, \ldots, n-1$ and their linear combinations. The coproduct is $\Delta(e_i) = e_i \otimes e_i$ (no summation over i). $(\Delta \otimes \mathrm{id}) \circ \Delta(e_i)$ is given by $e_i \otimes e_i \otimes e_i$ (no summation) and agrees with $(\mathrm{id} \otimes \Delta) \circ \Delta(e_i)$. All products $e_i e_j$ are contained in the set $\{e_i\}$, hence $\mathcal{H}$ is finite-dimensional, linearly spanned by $\{e_i\}$. The group law can then be given in terms of the constants $M_{ij}{}^k$ via $e_i e_j = M_{ij}{}^k e_k$. The role of derivatives is played by the difference operators $e_{ij} \equiv e_i - e_j$, $i < j$, with $\Delta(e_{ij}) = e_{ij} \otimes 1 + 1 \otimes e_{ij}$ and $\epsilon(e_{ij}) = 0$, $S(e_{ij}) = -e_{ij}$.

$\mathcal{A}$ is generated by the functions $\{f^i\}$, $i = 0, \ldots, n-1$, with $\langle e_i, f^j \rangle = \delta_i{}^j$, and is commutative. f^i is is a delta-like function peaked over e_i, hence $f^i f^j = 0$ whenever $i \neq j$ while each f^i squares to itself. Since no new functions can be produced by multiplication, $\mathcal{A}$ is linearly spanned by the set $\{f^i\}$. The coproduct in $\mathcal{A}$ is given by the same numerical constants that give the product in $\mathcal{H}$: $\Delta(f^i) = M_{jk}{}^i f^j f^k$. Notice that $(e_i \otimes f^i)(e_j \otimes f^j) = e_i \otimes e_i \otimes f^i$, which in turn is equal to $\Delta(e_i) \otimes f^i$. $e_i \otimes f^i$ is the *canonical element* and the above identity is in the spirit of $e^{a+b} = e^a e^b$.

The unit function is 1 on every e_i so that $\epsilon(e_i) = 1$. All f^i, except f^0, vanish on the identity hence $\epsilon(f^0) = 1$, $\epsilon(f^i) = 0$, $i \neq 0$. For the antipode we have $S(e_i) = e_i{}^{-1} \equiv S_i{}^j e_j$ and $S(f^i) = S_j{}^i f^j$. □

Example 2.2 *A discrete 'quantum space'*

$\mathcal{H}$ is generated by 1, x and y with $x^2 = 0$, $y^2 = y$ and $yx = -xy + x$. The coproduct is

$$\begin{aligned} \Delta(x) &= x \otimes 1 + 1 \otimes x - 2y \otimes x \\ \Delta(y) &= y \otimes 1 + 1 \otimes y - 2y \otimes y\,, \end{aligned} \tag{2.15}$$

i.e. x, y are of the difference operator type. The rest of the Hopf structure is $\epsilon(x) = \epsilon(y) = 0$ and $S(x) = -(1-2y)x$, $S(y) = y$ (notice that $S^2(x) = -x$). $\mathcal{H}$ is spanned linearly by $\{e_i\} = \{1, x, y, xy\}$. Writing $e_i e_j = (M_i)_j{}^k e_k$ and $\Delta(e_i) = (W^k)_i{}^j e_j e_k$ we find

$$M_0 = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} \qquad M_1 = \begin{pmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0&0&1&0 \\ 0&1&0&-1 \\ 0&0&1&0 \\ 0&0&0&0 \end{pmatrix} \quad M_3 = \begin{pmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix} \tag{2.16}$$

$$W_0 = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} \qquad W_1 = \begin{pmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \end{pmatrix}$$

$$W_2 = \begin{pmatrix} 0&0&0&0 \\ 0&-2&0&0 \\ 1&0&-2&0 \\ 0&0&0&0 \end{pmatrix} \quad W_3 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&-2&0 \end{pmatrix}.$$

The dual Hopf algebra $\mathcal{A}$ is spanned by $\{f^i\} = \{1, a, b, ab\}$, with commutation relations $a^2 = 0$, $b^2 = -2b$ and $ba = -ab - 2a$. The Hopf structure is

$$\begin{aligned} \Delta(a) &= a \otimes 1 + 1 \otimes a + b \otimes a \\ \Delta(b) &= b \otimes 1 + 1 \otimes b + b \otimes b \end{aligned} \tag{2.17}$$

and $\epsilon(a) = \epsilon(b) = 0$, $S(a) = a(1+b)$, $S(b) = b$ (so that $S^2(a) = -a$). □

Background material on Hopf algebras can be found in [12] and [9].

3. Quantum Integrals

3.1 A trace formula

Having codified translations in the coproduct, we now define a *right integral* in the Hopf algebra $\mathcal{A}$ as a map from functions to numbers, $\langle\cdot\rangle^R : \mathcal{A} \to \mathbb{C}$, which is invariant under right translations

$$\langle a_{(1)}\rangle^R a_{(2)} = \langle a\rangle^R 1_{\mathcal{A}} \qquad (3.1)$$

for all a in $\mathcal{A}$. We call $\langle\cdot\rangle^R$ *trivial* if all $\langle f^i\rangle^R$ are zero. *Left integrals* are similarly defined via

$$a_{(1)}\langle a_{(2)}\rangle^L = 1_{\mathcal{A}}\langle a\rangle^L \,. \qquad (3.2)$$

Radford and Larson [5] give the following trace formula for the right integral

$$\langle a\rangle^R_{tr} \equiv \left\langle S^2(e_i), f^i a\right\rangle \,, \qquad (3.3)$$

where $e_i \otimes f^i$ is the canonical element, as they warn though, it does not always produce a non-trivial result.

Example 3.1 *Integration on discrete groups*

Let's try (3.3) in a classical setting. $S^2 = \mathrm{id}$ in this case and $\langle a\rangle^R_{tr} = \left\langle e_i, f^i a\right\rangle = \sum_i f^i(e_i)a(e_i) = \sum_i a(e_i)$ which is the standard formula for the integral on the discrete group $\mathcal{H}$, normalized so that $\langle 1_{\mathcal{A}}\rangle^R_{tr} = |\mathcal{H}|$. □

Notice that (3.3) defines a (possibly trivial) invariant integral for *every* Hopf algebra, without requiring that it be of the function type. Consequently, we can use it to evaluate the integral of a group element or a difference operator. What is, classically, the meaning of such an integral? Just like the integral of a function is the sum of its pairings with all the elements of a basis in the dual, the integral of a group element is the sum of the values, on that particular element, of all the functions in a basis in the dual, and likewise for a difference operator.

Example 3.2 *Integral of a group element*

We use again (3.3), with $S^2 = \mathrm{id}$, to compute the integral of a group element e_i. We have $\langle e_i\rangle = \left\langle e_i e_j, f^j\right\rangle = {M_{ij}}^k \left\langle e_k, f^j\right\rangle = {M_{ij}}^j = {\delta_i}^0$, since from $e_i e_j = e_j$ it follows that $e_i = e_0 = 1$. For the difference operators e_{ij} we get then $\langle e_{ij}\rangle = \delta_{i0}$. □

Example 3.3 *Failure of the trace formula*

We compute the integral given by (3.3) for the Hopf algebra of Ex. 2.2. Using the definition of W^k given in the above example we find from (3.3)

$$\langle f^k\rangle^R_{tr} = \mathrm{Tr}(S^2 W^k) \,. \qquad (3.4)$$

Inspection of the matrices given explicitly in (2.16) shows that $\langle f^k\rangle^R_{tr} = 0$, $k = 0, \ldots, 3$. □

Further material for this section can be found in [6, 11, 13].

3.2 A non-trivial trace formula

We want to analyze now under what conditions does (3.3) fail and try, if possible, to modify it so that it furnishes always a non-trivial result. Our approach will be based on a 'vacuum expectation value' treatment of the integral [15]. To arrive at such a formulation, we first remark that the invariance relation (3.1), when paired with an arbitrary x in $\mathcal{H}$ gives

$$\langle x \triangleright a\rangle = \epsilon(x)\langle a\rangle \,, \qquad (3.5)$$

where $x \triangleright a \equiv a_{(1)} \left\langle x, a_{(2)}\right\rangle$. For x a group-like element, $x \triangleright a$ is the function a (right) translated by x. When x is a difference-like operator, $x \triangleright a$ is the 'derivative' of a along x. In the first case, $\epsilon(x) = 1$ and (3.5) states that the integral of the translated function is equal to that of the original one while in the second, $\epsilon(x) = 0$ and we get that the integral of a derivative is zero.

Given that x acts on products of functions via its coproduct, $x \triangleright (ab) = (x_{(1)} \triangleright a)(x_{(2)} \triangleright b)$, one can form a new algebra $\mathcal{A} \rtimes \mathcal{H}$, the *semidirect product* of $\mathcal{A}$ and $\mathcal{H}$, containing these as subalgebras and with cross relations

$$\begin{aligned} xa &= (x_{(1)} \triangleright a)x_{(2)} \\ &= a_{(1)} \left\langle x_{(1)}, a_{(2)}\right\rangle x_{(2)} \,. \end{aligned} \qquad (3.6)$$

The inverse relations are

$$ax = x_{(2)} \left\langle x_{(1)}, S^{-1}(a_{(2)}) \right\rangle a_{(1)} \,. \qquad (3.7)$$

Consider now two formal symbols $|\Omega_{\mathcal{H}}\rangle$ and $|\Omega_{\mathcal{A}}\rangle$, the $\mathcal{H}$ and $\mathcal{A}$-*right vacua* respectively [4], which satisfy

$$\begin{aligned} x|\Omega_{\mathcal{H}}\rangle &= \epsilon(x)|\Omega_{\mathcal{H}}\rangle \\ a|\Omega_{\mathcal{A}}\rangle &= \epsilon(a)|\Omega_{\mathcal{A}}\rangle. \end{aligned} \qquad (3.8)$$

Left vacua $\langle\Omega_{\mathcal{H}}|$, $\langle\Omega_{\mathcal{A}}|$ are defined analogously. In terms of these, the inner product $\langle x, a\rangle$ can be given as the 'expectation value'

$$\begin{aligned} \langle\Omega_{\mathcal{A}}|xa|\Omega_{\mathcal{H}}\rangle &= \langle\Omega_{\mathcal{A}}|a_{(1)} \left\langle x_{(1)}, a_{(2)} \right\rangle x_{(2)}|\Omega_{\mathcal{H}}\rangle \\ &= \langle x, a\rangle \,, \end{aligned}$$

if we normalize the vacua so that $\langle\Omega_{\mathcal{A}}|\Omega_{\mathcal{H}}\rangle = \langle\Omega_{\mathcal{H}}|\Omega_{\mathcal{A}}\rangle = 1$. Similarly, the left action of $\mathcal{H}$ on $\mathcal{A}$ can be written as

$$x \triangleright a|\Omega_{\mathcal{H}}\rangle = xa|\Omega_{\mathcal{H}}\rangle \,. \qquad (3.9)$$

We can now write a symbolic expression for our 'vacuum' integral

$$\langle a\rangle_v^R \sim |\Omega_{\mathcal{A}}\rangle\langle\Omega_{\mathcal{H}}|a|\Omega_{\mathcal{H}}\rangle\langle\Omega_{\mathcal{A}}|. \qquad (3.10)$$

Invariance in the form (3.5) follows from (3.9) and the left version of the first of (3.8). This can be turned into something more than symbolic if we borrow from [3] the result that the operators $|\Omega_{\mathcal{A}}\rangle\langle\Omega_{\mathcal{H}}|$, $|\Omega_{\mathcal{H}}\rangle\langle\Omega_{\mathcal{A}}|$ can be represented in $\mathcal{A}\rtimes\mathcal{H}$ as

$$\begin{aligned} |\Omega_{\mathcal{H}}\rangle\langle\Omega_{\mathcal{A}}| &\sim E \equiv S^{-1}(f^i)e_i \,, \\ |\Omega_{\mathcal{A}}\rangle\langle\Omega_{\mathcal{H}}| &\sim \bar{E} \equiv S^2(e_i)f^i \,. \end{aligned} \qquad (3.11)$$

Indeed, one easily finds that $Ea = \epsilon(a)E$, $xE = \epsilon(x)E$ *etc.*, as well as that $E^2 = E$, $\bar{E}^2 = \bar{E}$. Then (3.10) can be given a precise meaning by defining [2]

$$\langle a\rangle_v^R |\Omega_{\mathcal{A}}\rangle\langle\Omega_{\mathcal{A}}| \equiv \bar{E}aE \,. \qquad (3.12)$$

This needs perhaps some explanation. The r.h.s. above is an element of $\mathcal{A}\rtimes\mathcal{H}$ that realizes the operator $|\Omega_{\mathcal{A}}\rangle\langle\Omega_{\mathcal{A}}|$. The latter is unique up to scale, and hence, for different inputs a in the l.h.s., we get numerical multiples of the same operator in the r.h.s. (after using the commutation relations in $\mathcal{A}\rtimes\mathcal{H}$ to bring the r.h.s. in some standard ordering). After choosing a normalization for $|\Omega_{\mathcal{A}}\rangle\langle\Omega_{\mathcal{A}}|$, we may define $\langle a\rangle_v^R$ to be the numerical constant that multiplies $|\Omega_{\mathcal{A}}\rangle\langle\Omega_{\mathcal{A}}|$ in the l.h.s. above. Notice that when the left and right integrals coincide, $\bar{E}aE$ is a pure function and $|\Omega_{\mathcal{A}}\rangle\langle\Omega_{\mathcal{A}}|$ is realized in $\mathcal{A}$. The connection of this definition with the trace formula is revealed by computing

$$\begin{aligned} \bar{E}aE &= S^2(e_i)f^i aE \\ &= f^i_{(1)}a_{(1)} \left\langle S^2(e_i), f^i_{(2)}a_{(2)} \right\rangle E \\ &= f^n \left\langle e_n, f^i_{(1)}a_{(1)} \right\rangle \left\langle S^2(e_i), f^i_{(2)}a_{(2)} \right\rangle E \\ &= f^n \left\langle e_n S^2(e_i), f^i a \right\rangle E \,. \end{aligned} \qquad (3.13)$$

The quantity multiplying E in the last line above is a modified trace which is both right-invariant and non-trivial. Indeed, invariance is easily verified along the lines of the proof of the original trace formula, Eq. (3.3). For the non-triviality, we set $\Theta^i_l \equiv \left\langle e_i S^2(e_k), f^k f^l \right\rangle$ and compute

$$\begin{aligned} S^2(e_i)f^i &= f^i_{(1)} \left\langle S^2(e_{i_{(1)}}), f^i_{(2)} \right\rangle S^2(e_{i_{(2)}}) \\ &= f^i_{(1)}f^j_{(1)} \left\langle S^2(e_i), f^i_{(2)}f^j_{(2)} \right\rangle S^2(e_j) \\ &= S^2(f^i)f^j \left\langle e_i S^2(e_k), f^k f^l \right\rangle \\ &\qquad S^2(e_j)S^2(e_l) \\ &= \Theta^l_i S^2(f^i)f^j S^2(e_j)S^2(e_l) \end{aligned} \qquad (3.14)$$

from which we may conclude that not all Θ^l_i are zero, since $S^2(e_i)f^i \neq 0$ (see [13] for an alternative proof). Some Hopf algebra trickery shows that $\left\langle xS^2(e_i), f^i a \right\rangle$ supplies actually a left integral for x (as well as a right one for a), so it is proportional to their product. Then (3.3) returns the integral of a multiplied by the integral of the unit element in the dual and will hence fail whenever the latter vanishes. Our modified trace above avoids this by summing over all integrals in the dual and 'tagging' each by the (linearly independent) f^n. The appearance of $|\Omega_{\mathcal{A}}\rangle\langle\Omega_{\mathcal{A}}|$ in (3.12) might look a little strange now, but it is actually a blessing in disguise. When later we look for an integral in braided Hopf algebras, it will become apparent that a purely numerical integral cannot, in general, transform covariantly and the operator $|\Omega_{\mathcal{A}}\rangle\langle\Omega_{\mathcal{A}}|$ that the braided analog of the above construction produces exactly compensates for the fact that we do not explicitly display the measure used in our integrations.

Example 3.4 *The modified trace*

Continuing Ex. 2.2, we derive the commutation relations of (3.6)

$$\begin{aligned} xa &= 1+b+ax \\ xb &= -(b+2)x \\ ya &= ay \\ yb &= 1+b-(b+2)y\,. \end{aligned} \tag{3.15}$$

We also compute the vacuum projectors in (3.11)

$$\begin{aligned} E &= 1-ax(1-2y)+by-abx(1-y) \\ \bar{E} &= 1-xa+yb-xyab\,. \end{aligned} \tag{3.16}$$

One easily verifies that they satisfy indeed $Ea = \epsilon(a)E$ *etc.*. A straightforward calculation now gives

$$\begin{aligned} \langle 1_{\mathcal{A}}\rangle_v^R\,|\Omega_{\mathcal{A}}\rangle\langle\Omega_{\mathcal{A}}| &= \bar{E}E = 0 \\ \langle a\rangle_v^R\,|\Omega_{\mathcal{A}}\rangle\langle\Omega_{\mathcal{A}}| &= \bar{E}aE = -abg \\ \langle b\rangle_v^R\,|\Omega_{\mathcal{A}}\rangle\langle\Omega_{\mathcal{A}}| &= \bar{E}bE = 0 \\ \langle ab\rangle_v^R\,|\Omega_{\mathcal{A}}\rangle\langle\Omega_{\mathcal{A}}| &= \bar{E}abE = abg \end{aligned} \tag{3.17}$$

where $g \equiv 1-2y$. The element $abg \in \mathcal{A} \rtimes \mathcal{H}$ realizes (up to scale) the operator $|\Omega_{\mathcal{A}}\rangle\langle\Omega_{\mathcal{A}}|$. We fix the normalization so that $\langle a\rangle_v^R = -1$ and $\langle ab\rangle_v^R = 1$, the other two integrals being zero. We can also treat E above as a spectator to find $\bar{E}aE = -abE$ and $\bar{E}abE = abE$, thus isolating the right delta function $\delta_{\mathcal{A}}^R = ab$. A left invariant integral in $\mathcal{A}$ is given by $\langle\cdot\rangle_v^L = \langle S(\cdot)\rangle_v^R$. We find $\langle ab\rangle_v^L = -1$, all other integrals being zero.

Noting that $x \triangleleft a \equiv \langle x_{(1)}, a\rangle\, x_{(2)}$ satisfies $Ex \triangleleft a = Exa$, we may compute a *left*-invariant integral in $\mathcal{H}$ via

$$\langle z\rangle_v^L|\Omega_{\mathcal{H}}\rangle\langle\Omega_{\mathcal{H}}| = Ez\bar{E} \qquad z \in \mathcal{H}\,. \tag{3.18}$$

We find that the only non-zero integral is

$$\begin{aligned} \langle z\rangle_v^L|\Omega_{\mathcal{H}}\rangle\langle\Omega_{\mathcal{H}}| &= Exy\bar{E} \\ &= -(1+b)x(1-y)\,; \end{aligned} \tag{3.19}$$

notice that $\langle z\rangle^L \propto \langle z, \delta_{\mathcal{A}}^R\rangle$. The r.h.s. of (3.19) realizes $|\Omega_{\mathcal{H}}\rangle\langle\Omega_{\mathcal{H}}|$ in $\mathcal{A} \rtimes \mathcal{H}$. A right integral in $\mathcal{H}$ is given, as before, via the antipode. □

4. Hopf Algebras + Statistics = Braided Hopf Algebras

4.1 The universal R-matrix

It is a rule of thumb in algebra that the interesting way to generalize a symmetry or constraint is by relaxing it 'up to similarity'. Deforming the coproduct of our 'points' into a non-cocommutative one gives rise to a rich algebraic structure when the two coproducts Δ and $\Delta' \equiv \tau\circ\Delta$ (with τ the permutation map) are related by conjugation

$$\Delta'(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1} \tag{4.1}$$

for all x in $\mathcal{H}$. Here $\mathcal{R} \in \mathcal{H}\otimes\mathcal{H}$ is the *universal R-matrix* which satisfies additionally

$$(\Delta\otimes\mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} \tag{4.2}$$

$$(\mathrm{id}\otimes\Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}\,. \tag{4.3}$$

A Hopf algebra $\mathcal{H}$ for which an $\mathcal{R}$ exists is called *quasitriangular*. Consider the collection of (left) representation spaces of such an $\mathcal{H}$ (left *$\mathcal{H}$-modules*). These are vector spaces (possibly with additional structure, like product, coproduct *etc.*) on which $\mathcal{H}$ acts from the left. This collection is equipped with a *tensor product* operation that combines two $\mathcal{H}$-modules V, W to form a new one, their *tensor product* $V\otimes W$, on which $\mathcal{H}$ acts via its coproduct

$$h \triangleright (v\otimes w) = (h_{(1)}\triangleright v)\otimes(h_{(2)}\triangleright w)\,. \tag{4.4}$$

Another way to obtain new modules from old ones, at least in the classical case, is via the transposition $\tau : V\otimes W \mapsto W\otimes V$. This operation commutes, in the classical case, with the action of $\mathcal{H}$ since $\Delta' = \Delta$. To obtain an analogous operation in the Hopf algebra case, we need to employ a *braided transposition* Ψ which consists in first acting with $\mathcal{R}$ on the tensor product of the modules to be transposed and then effecting τ

$$\begin{aligned} \Psi(v\otimes w) &\equiv \tau(\mathcal{R}^{(1)}\triangleright v\otimes\mathcal{R}^{(2)}\triangleright w) \\ &= \mathcal{R}^{(2)}\triangleright w\otimes\mathcal{R}^{(1)}\triangleright v\,, \end{aligned} \tag{4.5}$$

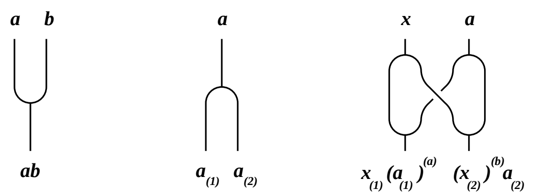
where we have written $\mathcal{R} \equiv \mathcal{R}^{(1)}\otimes\mathcal{R}^{(2)}$ (summation implied). Things get interesting when $\mathcal{R}' \equiv \tau(\mathcal{R}) = \mathcal{R}^{(2)}\otimes\mathcal{R}^{(1)} \neq \mathcal{R}^{-1}$, which is often the case. This results in $\Psi^2 \neq \mathrm{id}$ (by

Ψ^2 we mean $\Psi_{V\otimes W} \circ \Psi_{W\otimes V}$). It is as though the objects we transpose were hanging from the ceiling by threads and successive transpositions were recorded in the entanglement of the threads. Taking this picture seriously, we will adopt a diagrammatic notation in which Ψ, Ψ^{-1} look like

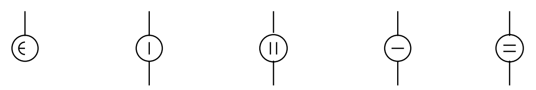
where $\Psi(v \otimes w) \equiv w^{(a)} \otimes v^{(b)}$ and $\Psi^{-1}(v \otimes w) \equiv w^{(a')} \otimes v^{(b')}$. In other words, we imagine the algebra elements flowing from top to bottom along the braids, with over- and under-crossings representing the effect of Ψ and Ψ^{-1} respectively. We see that the existence of an $\mathcal{R}$ in $\mathcal{H}$ endows all algebraic structures covariant under the action of $\mathcal{H}$ with a natural *statistics*. The usual bosonic and fermionic rules for transposition can be put in this language but, in general, Ψ can be much more drastic, as the action of $\mathcal{R}$ that precedes that of τ if often far from trivial.

4.2 Braided Hopf algebras

One might think now of developing an algebra, involving products, coproducts *etc.*, in which all typographical transpositions are effected by Ψ, Ψ^{-1}, rather than τ. *Braided Hopf algebras* are a transcription of ordinary Hopf algebras in this braided setting [10]. Thus, one has a braided antipode, braided coproduct *etc.*, which satisfy braided versions of the standard Hopf algebra axioms. Denoting *e.g.* the product and coproduct by the first two of the vertices below



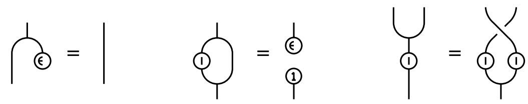
one gets the braided version of the multiplicativity of the coproduct expressed by the third diagram above. Similarly, we denote ϵ, S, S^2, S^{-1}, S^{-2} respectively by



and express the braided analogues of *e.g.*

$$
\begin{aligned}
a_{(1)}\epsilon(a_{(2)}) &= a \\
S(a_{(1)})a_{(2)} &= \epsilon(a) \\
S(ab) &= S(b)S(a)
\end{aligned}
$$

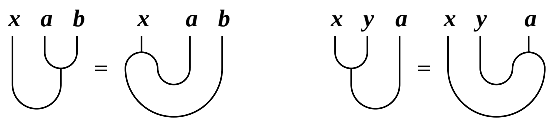
by the diagrams



respectively (the output braid of the counit is supressed since, being a number, it braids trivially). Covariance is codified in the requirement that one should be able to move crossings past all vertices and boxes, *e.g.* the relations

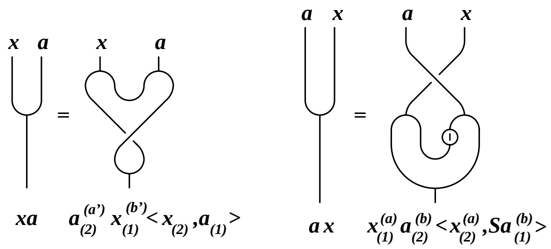
should hold. The inner product and the canonical element look like

For the product-coproduct duality we adopt a convention which is slightly different from the one we used before, as shown below



As a result, to get the unbraided expression that corresponds to any of the diagrammatic identities that appear in the following, one should translate the diagrams, ignoring the braiding information, into the language of Sect. 3 and then set $\Delta \to \Delta'$, $S \to S^{-1}$. Notice that all diagrams reveal new (dual) information when viewed upside down. The commutation relations in the semidirect product (*i.e.* the braided analogue of

(3.6)) are



A detailed exposition of the basics (and more) of braided Hopf algebras can be found in [10] and references therein.

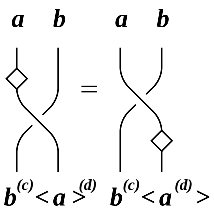
5. Braided Integrals

5.1 Problems with braiding

In the case of braided Hopf algebras, the integral presents an additional complication. Defining it, along the lines of (3.1), as a number, implies that it braids trivially, regardless of the transformation properties of the integrand. This can easily be seen to lead to problems when combined with the translational invariance property, as the following toy example shows.

Example 5.1 *The fermionic line*

Consider the algebra $\mathcal{A}$ of functions on the classical fermionic line. It is generated by 1 and a fermionic variable ξ, with $\xi^2 = 0$, and admits the braided coproduct $\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi$, with Ψ the standard fermionic braiding, $\Psi(\xi \otimes \xi) = -\xi \otimes \xi$ ($\Psi^2 = \mathrm{id}$) in this case). $\epsilon(\xi) = 0$ and $S(\xi) = -\xi$. Representing the integral with a rhombus, we want it to satisfy



(5.1)

Taking both inputs to be ξ, and using the Berezin result $\langle \xi \rangle = 1$, we get for the l.h.s.

$$\begin{aligned} \Psi(\langle \xi \rangle \otimes \xi) &= \Psi(1 \otimes \xi) \\ &= \xi \otimes 1 \end{aligned}$$

while the r.h.s. gives

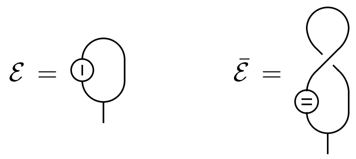
$$\begin{aligned} (\mathrm{id} \otimes \langle \cdot \rangle) \circ \Psi(\xi \otimes \xi) &= -(\mathrm{id} \otimes \langle \cdot \rangle)(\xi \otimes \xi) \\ &= -\xi \otimes 1 \,. \end{aligned}$$

The problem originates in the absense of any explicit mention, in our algebraic treatment, to the 'measure' $d\xi$. □

More material on braided integrals is in [1, 7, 8].

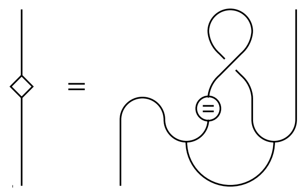
5.2 The braided vacuum projectors

As we mentioned at the end of Sec. 3, the 'vacuum expectation value' approach suggests a solution [2]. Our starting point is the braided analogue of the vacuum projectors of (3.11). Denoting them by $\mathcal{E}$, $\bar{\mathcal{E}}$ we find



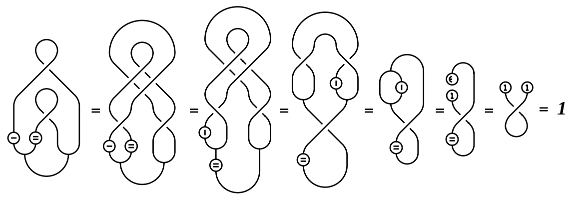
(5.2)

We give, in Fig. 1, the proof that $\mathcal{E}a = \mathcal{E}\epsilon(a)$, for all a in $\mathcal{A}$—the rest of the required relations are proved similarly. Forming $\bar{\mathcal{E}}a\mathcal{E}$ and treating $\mathcal{E}$ as a spectator, we get a glimpse of the inner workings of the rhombus



(5.3)

The output in the above diagram is a multiple of the delta function, the coefficient being the numerical integral of the input. Again, one can show that this provides a non-trivial integral for all finite dimensional braided Hpf algebras, transcribing either (3.14) or the more elegant proof in [13]. Opting for the latter we find



(5.4)

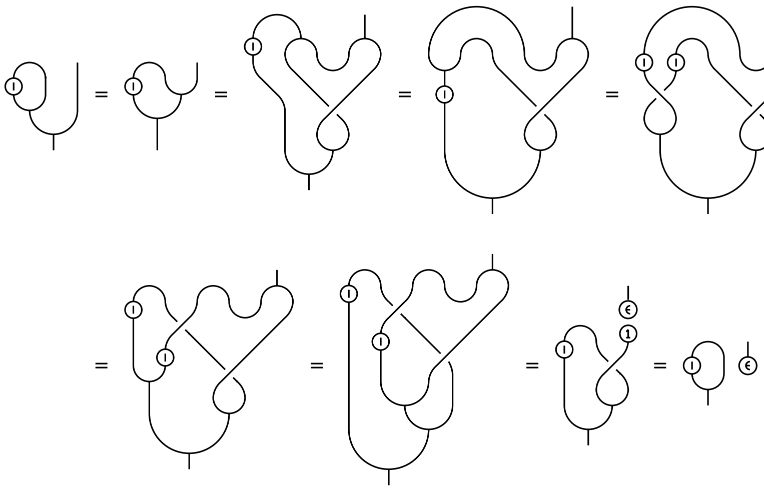
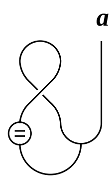


Figure 1: Proof of $\mathcal{E}a = \mathcal{E}\epsilon(a)$, $a \in \mathcal{A}$

from which non-triviality follows. The analogue of (3.3) is easily deduced from (5.3)

$\langle a \rangle^{R}_{tr} =$ 

(5.5)

and a direct proof of its invariance is shown in Fig. 2.

Example 5.2 *The Berezin integral as a sum*

Continuing our toy example 5.1, we introduce a fermionic derivative σ, with $\sigma^2 = 0$ and $\sigma\xi = 1 - \xi\sigma$ and the standard Hopf structure and braiding. Then, $\{e_i\} = \{1, \sigma\}$ and $\{f^i\} = \{1, \xi\}$ which gives $\mathcal{E} = 1 - \xi\sigma = \sigma\xi$ and $\bar{\mathcal{E}} = 1 - \sigma\xi = \xi\sigma$. For the integrals we compute $\bar{\mathcal{E}}\mathcal{E} = 0$ and $\bar{\mathcal{E}}\xi\mathcal{E} = \xi$. Left and right integrals coincide in this case so $|\Omega_{\mathcal{A}}\rangle\langle\Omega_{\mathcal{A}}|$ is realized in $\mathcal{A}$ by ξ and we recover the Berezin result. Notice that (5.3) gives the Berezin integral as a sum over 'points'. The integral of the unit function receives two equal (unit) contributions from the two 'points' in the space, but the undercrossing in $\bar{\mathcal{E}}$ flips the sign of one of them so that they cancel. □

Example 5.3 *The quantum fermionic plane*

This was introduced in [14, 16]—we follow the conventions in [2]. $\mathcal{A}$ is generated by the fermionic coordinate functions ξ_i, $i = 1, \ldots, N$. They satisfy

$$\xi_2\xi_1 = -q\hat{R}_{12}\xi_2\xi_1 \tag{5.6}$$

and are dual to the derivatives σ_i, $i = 1, \ldots, N$ that generate $\mathcal{H}$ with relations

$$\sigma_1\sigma_2 = -q\sigma_1\sigma_2\hat{R}_{12}\,. \tag{5.7}$$

In $\mathcal{A} \rtimes \mathcal{H}$ we have

$$\sigma_i\xi_j = \delta_{ij} - q\hat{R}^{-1}_{mj,ni}\xi_n\sigma_m\,. \tag{5.8}$$

The braided coproduct is $\Delta(\xi_i) = \xi_i \otimes 1 + 1 \otimes \xi_i$ and similarly for the σ's. The braiding is given by

$$\begin{aligned}
\Psi(\xi_2 \otimes \xi_1) &= -q\hat{R}_{12}\xi_2 \otimes \xi_1 \\
\Psi(\sigma_1 \otimes \sigma_2) &= -q\sigma_1 \otimes \sigma_2\hat{R}^{-1}_{12} \\
\Psi(\xi_i \otimes \sigma_j) &= -q^{-1}D_{la}\hat{R}_{ia,bk}D^{-1}_{bj}\sigma_l \otimes \xi_k \\
\Psi(\sigma_i \otimes \xi_j) &= -q^{-1}\hat{R}_{kj,li}\xi_l \otimes \sigma_k\,.
\end{aligned}$$

For the canonical element we find

$$f^i \otimes e_i = e_{q^{-1}}(\xi_i \otimes \sigma_i)\,,$$

(compare with the bosonic quantum plane result in [4]) where

$$e_q(x) = \textstyle\sum_{k=0}^{\infty} \frac{1}{[k]_q!}x^k\,, \qquad [k]_q = \frac{1-q^{2k}}{1-q^2}\,,$$

$$[k]_q! = [1]_q[2]_q \ldots [k]_q\,, \qquad [0]_q \equiv 1\,.$$

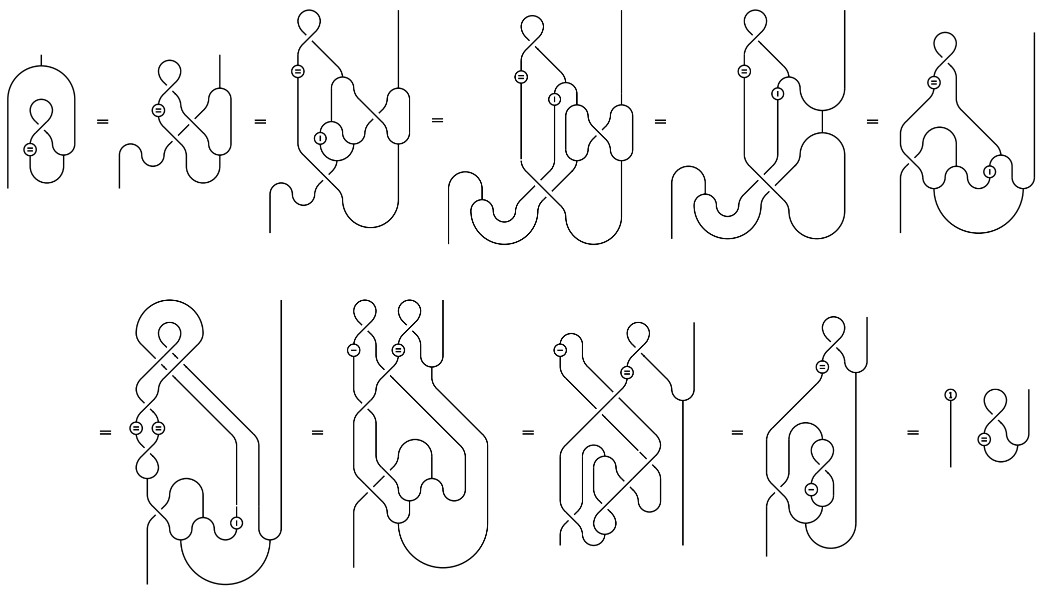


Figure 2: Proof of the invariance of the integral

The vacuum projectors are

$$\mathcal{E} = \sum_{k=0}^{N} \frac{(-1)^k}{[k]_{q^{-1}}!} \xi_{i_{1\ldots k}} \sigma_{i_{k\ldots 1}}$$

$$\bar{\mathcal{E}} = \sum_{k=0}^{N} \frac{(-1)^k q^k}{[k]_q!} D_{i_1 j_1} \ldots D_{i_k j_k} \sigma_{i_{k\ldots 1}} \xi_{j_{1\ldots k}} \, ,$$

where $\xi_{i_{1\ldots k}} \equiv \xi_{i_1} \ldots \xi_{i_k}$ and similarly for σ. The integral of a monomial $\xi_{i_{1\ldots r}}$, $r < N$, is given by

$$\bar{\mathcal{E}} \xi_{i_{1\ldots r}} \mathcal{E} = \left(\sum_{k=0}^{A} \frac{(-1)^k q^{k(k-2A+1)} [A]_q!}{[k]_q! [A-k]_q!} \right) \xi_{i_{1\ldots r}} \mathcal{E} \, ,$$

with $A \equiv N - r$. Using standard q-machinery, the sum in parentheses can be shown to vanish for $0 \leq r < N$ while for $r = N$ the integrand is (proportional to) the (left- and right-) delta function. Then both $\bar{\mathcal{E}}$ and $\mathcal{E}$ reduce to $1_{\mathcal{A} \rtimes \mathcal{H}}$ and the numerical integral is (proportional) to 1. We conclude that the integral is essentially independent of q. Notice that, in the limit $q \to -1$, the algebra (5.6) *does not* become a bosonic one, *e.g.* (5.6) implies $\xi_i^2 = 0$ which persists for all values of q. □

Acknowledgments

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