Differential calculi on finite groups

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Abstract: A brief review of bicovariant differential calculi on finite groups is given, with some new developments on diffeomorphisms and integration. We illustrate the general theory with the example of the nonabelian finite group S_3.

1. Introduction

Differential calculi can be constructed on spaces that are more general than differentiable manifolds. Indeed the algebraic construction of differential calculus in terms of Hopf structures allows to extend the usual differential geometric quantities (connection, curvature, metric, vielbein etc.) to a variety of interesting spaces that include quantum groups, noncommutative spacetimes (i.e. quantum cosets), and discrete spaces.

In this contribution we concentrate on the differential geometry of finite group "manifolds". As we will discuss, these spaces can be visualized as collections of points, corresponding to the finite group elements, and connected by oriented links according to the particular differential calculus we build on them. Although functions $f \in Fun(G)$ on finite groups G commute, the calculi that are constructed on $Fun(G)$ by algebraic means are in general noncommutative, in the sense that differentials do not commute with functions, and the exterior product does not coincide with the usual antisymmetrization of the tensor product.

Among the physical motivations for finding differential calculi on finite groups we mention the possibility of using finite group spaces as internal spaces for Kaluza-Klein compactifications of supergravity or superstring theories (for example Connes' reconstruction of the standard model in terms of noncommutative geometry [1] can be recovered as Kaluza-Klein compactification of Yang-Mills theory on an appropriate discrete internal space). Differential calculi on discrete spaces can be of use in the study of integrable models, see for ex. ref. [2]. Finally gauge and gravity theories on finite group spaces may be used as lattice approximations. For example the action for pure Yang-Mills $\int F \wedge {}^*F$ considered on the finite group space $Z^N \times Z^N \times Z^N \times Z^N$, yields the usual Wilson action of lattice gauge theories, and $N \to \infty$ gives the continuum limit [3]. New lattice theories can be found by choosing different finite groups.

A brief review of the differential calculus on finite groups is presented. Most of this material is not new, and draws on the treatment of ref.s [5, 6, 7], where the Hopf algebraic approach of Woronowicz [8] for the construction of differential calculi is adapted to the setting of finite groups. Some developments on Lie derivative, diffeomorphisms and integration are new. The general theory is illustrated in the case of S_3.

2. Differential calculus on finite groups

Fun(G) as a Hopf algebra

Let G be a finite group of order n with generic element g and unit e. Consider $Fun(G)$, the set of complex functions on G. An element f of $Fun(G)$ is specified by its values $f_g \equiv f(g)$ on the group elements g, and can be written as

$$f = \sum_{g \in G} f_g x^g, \quad f_g \in \boldsymbol{C} \tag{2.1}$$

where the functions x^g are defined by

$$x^g(g') = \delta^g_{g'} \tag{2.2}$$

Thus $Fun(G)$ is a n-dimensional vector space, and the n functions x^g provide a basis. $Fun(G)$ is also a commutative algebra, with the usual pointwise sum and product $[(f+h)(g) = f(g) + h(g), (f \cdot h)(g) = f(g)h(g), (\lambda f)(g) = \lambda f(g), f, h \in Fun(G), \lambda \in \boldsymbol{C}]$ and unit I defined by $I(g) = 1, \forall g \in G$. In particular:

$$x^g x^{g'} = \delta_{g,g'} x^g, \qquad \sum_{g \in G} x^g = I \tag{2.3}$$

The G group structure induces a Hopf algebra structure on $Fun(G)$, with coproduct Δ, coinverse κ and counit ε defined by group multiplication, inverse and unit as:

$$\Delta(f)(g, g') = f(gg'), \tag{2.4}$$
$$\kappa(f)(g) = f(g^{-1}), \tag{2.5}$$
$$\varepsilon(f) = f(e), \tag{2.6}$$

On the basis functions x^g the costructures take the form:

$$\Delta(x^g) = \sum_{h \in G} x^h \otimes x^{h^{-1}g}, \ \kappa(x^g) = x^{g^{-1}}, \ \varepsilon(x^g) = \delta^g_e \tag{2.7}$$

The coproduct is related to the pullback induced by left or right multiplication of G on itself. Consider the left multiplication by g_1:

$$L_{g_1} g_2 = g_1 g_2, \quad \forall g_1, g_2 \in G \tag{2.8}$$

This induces the left action (pullback) $\mathcal{L}_{g_1}$ on $Fun(G)$:

$$\mathcal{L}_{g_1} f(g_2) \equiv f(g_1 g_2)|_{g_2}, \quad \mathcal{L}_{g_1} : Fun(G) \to Fun(G) \tag{2.9}$$

where $f(g_1 g_2)|_{g_2}$ means $f(g_1 g_2)$ seen as a function of g_2. For the basis functions we find easily:

$$\mathcal{L}_{g_1} x^g = x^{g_1^{-1} g} \tag{2.10}$$

Introducing the mapping $\mathcal{L} : Fun(G) \to Fun(G \times G) \approx Fun(G) \otimes Fun(G)$:

$$(\mathcal{L}f)(g_1, g_2) \equiv (\mathcal{L}_{g_1} f)(g_2) = f(g_1 g_2)|_{g_2} \tag{2.11}$$

we see that

$$\mathcal{L} = \Delta \tag{2.12}$$

Thus the coproduct mapping Δ on the function f encodes the information on all the left actions $\mathcal{L}_g, g \in G$ applied to f, without reference to a particular g ("point of the group manifold"). It also encodes the information on right actions.

Indeed one can define the right action $\mathcal{R}$ on $Fun(G)$ as:

$$(\mathcal{R}f)(g_1, g_2) \equiv (\mathcal{R}_{g_1} f)(g_2) = f(g_2 g_1)|_{g_2} \tag{2.13}$$

Introducing the flip operator $\tau : Fun(G \times G) \to Fun(G \times G)$:

$$(\tau u)(g_1, g_2) \equiv u(g_2, g_1), \quad u \in Fun(G \times G) \tag{2.14}$$

it is easy to find that:

$$\mathcal{R} = \tau \circ \Delta \tag{2.15}$$

For the basis functions:

$$\mathcal{R}_{g_1} x^g = x^{g g_1^{-1}}, \ \mathcal{R} x^g = \tau \circ \Delta(x^g) = \sum_{h \in G} x^{h^{-1} g} \otimes x^h \tag{2.16}$$

Finally:

$$\mathcal{L}_{g_1} \mathcal{L}_{g_2} = \mathcal{L}_{g_1 g_2}, \quad \mathcal{R}_{g_1} \mathcal{R}_{g_2} = \mathcal{R}_{g_2 g_1}, \tag{2.17}$$
$$\mathcal{L}_{g_1} \mathcal{R}_{g_2} = \mathcal{R}_{g_2} \mathcal{L}_{g_1} \tag{2.18}$$

Bicovariant differential calculus

Differential calculi can be constructed on Hopf algebras A by algebraic means, using the costructures of A [8]. In the case of finite groups G, differential calculi on $A = Fun(G)$ have been discussed in ref.s [5, 6, 7]. Here we give the main results derived in [7], to which we refer for a more detailed treatment.

A *first-order differential calculus* on A is defined by

i) a linear map d: $A \to \Gamma$, satisfying the Leibniz rule

$$d(ab) = (da)b + a(db), \quad \forall a, b \in A; \tag{2.19}$$

The "space of 1-forms" Γ is an appropriate bimodule on A, which essentially means that its elements can be multiplied on the left and on the right by elements of A [more precisely A is a left module if $\forall a, b \in A, \forall \rho, \rho' \in \Gamma$ we have: $a(\rho+\rho') = a\rho + a\rho'$, $(a+b)\rho = a\rho + b\rho$, $a(b\rho) = (ab)\rho$, $I\rho = \rho$. Similarly one defines a right module. A left and right module is a *bimodule* if $a(\rho b) = (a\rho)b$]. From the Leibniz rule $da = d(Ia) = (dI)a + Ida$ we deduce $dI = 0$.

ii) the possibility of expressing any $\rho \in \Gamma$ as

$$\rho = \sum_k a_k db_k \tag{2.20}$$

for some a_k, b_k belonging to A.

Left and right covariance

A differential calculus is *left or right covariant* if the left or right action of G ($\mathcal{L}_g$ or $\mathcal{R}_g$) commutes with the exterior derivative d. Requiring left and right covariance in fact *defines* the action of $\mathcal{L}_g$ and $\mathcal{R}_g$ on differentials: $\mathcal{L}_g db \equiv d(\mathcal{L}_g b), \forall b \in Fun(G)$ and similarly for $\mathcal{R}_g db$. More generally, on elements of Γ (one-forms) we define $\mathcal{L}_g$ as:

$$\mathcal{L}_g(adb) \equiv (\mathcal{L}_g a)\mathcal{L}_g db = (\mathcal{L}_g a) d(\mathcal{L}_g b) \tag{2.21}$$

and similar for $\mathcal{R}_g$. Computing for example the left and right action on the differentials dx^g yields:

$$\mathcal{L}_g(dx^{g_1}) \equiv d(\mathcal{L}_g x^{g_1}) = dx^{g^{-1}g_1}, \tag{2.22}$$
$$\mathcal{R}_g(dx^{g_1}) \equiv d(\mathcal{R}_g x^{g_1}) = dx^{g_1 g^{-1}} \tag{2.23}$$

A differential calculus is called *bicovariant* if it is both left and right covariant.

Left invariant one forms

As in usual Lie group manifolds, we can introduce a basis in Γ of left-invariant one-forms θ^g:

$$\theta^g \equiv \sum_{h\in G} x^{hg} dx^h \quad (= \sum_{h\in G} x^h dx^{hg^{-1}}), \tag{2.24}$$

It is immediate to check that $\mathcal{L}_k \theta^g = \theta^g$. The relations (2.24) can be inverted [as they should, since property ii) of a first order differential calculus must hold]:

$$dx^h = \sum_{g\in G} (x^{hg} - x^h)\theta^g \tag{2.25}$$

From $0 = dI = d\sum_{g\in G} x^g = \sum_{g\in G} dx^g = 0$ one finds:

$$\sum_{g\in G} \theta^g = \sum_{g,h\in G} x^h dx^{hg^{-1}} = \sum_{h\in G} x^h \sum_{g\in G} dx^{hg^{-1}} = 0 \tag{2.26}$$

Therefore we can take as basis of the cotangent space Γ the $n-1$ linearly independent left-invariant one-forms θ^g with $g \neq e$ (but smaller sets of θ^g can be consistently chosen as basis, see later).

The commutations between the basic 1-forms θ^g and functions $f \in Fun(G)$ are given by:

$$f\theta^g = \theta^g \mathcal{R}_g f \tag{2.27}$$

Thus functions do commute between themselves (i.e. $Fun(G)$ is a commutative algebra) but do not commute with the basis of one-forms θ^g. In this sense the differential geometry of $Fun(G)$ is noncommutative, the noncommutativity being milder than in the case of quantum groups $Fun_q(G)$(which are noncommutative algebras).

The right action of G on the elements θ^g is given by:

$$\mathcal{R}_h \theta^g = \theta^{ad(h)g}, \quad \forall h \in G \tag{2.28}$$

where ad is the adjoint action of G on G, i.e. $ad(h)g \equiv hgh^{-1}$. Then *bicovariant calculi are in 1-1 correspondence with unions of conjugacy classes (different from $\{e\}$)* [5]: if θ^g is set to zero, one must set to zero all the $\theta^{ad(h)g}$, $\forall h \in G$ corresponding to the whole conjugation class of g.

We denote by G' the subset corresponding to the union of conjugacy classes that characterizes the bicovariant calculus on G ($G' = \{g \in G | \theta^g \neq 0\}$). Unless otherwise indicated, repeated indices are summed on G' in the following.

A bi-invariant (i.e. left and right invariant) one-form Θ is obtained by summing on all θ^g with $g \neq e$:

$$\Theta = \sum_{g\neq e} \theta^g = \sum_{g\neq e} \omega^g \tag{2.29}$$

Exterior product

For a bicovariant differential calculus on a Hopf algebra A an *exterior product*, compatible with the left and right actions of G, can be defined by

$$\theta^{g_1} \wedge \theta^{g_2} = \theta^{g_1} \otimes \theta^{g_2} - \theta^{g_1^{-1} g_2 g_1} \otimes \theta^{g_1} \quad (2.30)$$

where the tensor product between elements $\rho, \rho' \in \Gamma$ is defined to have the properties $\rho a \otimes \rho' = \rho \otimes a\rho'$, $a(\rho \otimes \rho') = (a\rho) \otimes \rho'$ and $(\rho \otimes \rho')a = \rho \otimes (\rho' a)$.

Note that:

$$\theta^g \wedge \theta^g = 0 \qquad \text{(no sum on } g) \quad (2.31)$$

Left and right actions on $\Gamma \otimes \Gamma$ are simply defined by:

$$\mathcal{L}_h(\rho \otimes \rho') = \mathcal{L}_h \rho \otimes \mathcal{L}_h \rho', \quad (2.32)$$
$$\mathcal{R}_h(\rho \otimes \rho') = \mathcal{R}_h \rho \otimes \mathcal{R}_h \rho' \quad (2.33)$$

(with the obvious generalization to $\Gamma \otimes ... \otimes \Gamma$) so that for example:

$$\mathcal{L}_h(\theta^i \otimes \theta^j) = \theta^i \otimes \theta^j, \quad (2.34)$$
$$\mathcal{R}_h(\theta^i \otimes \theta^j) = \theta^{ad(h)i} \otimes \theta^{ad(h)j} \quad (2.35)$$

We can generalize the definition (2.37)to exterior products of n one-forms:

$$\theta^{i_1} \wedge ... \wedge \theta^{i_n} \equiv W^{i_1 i_2}{}_{j_1 k_1} W^{k_1 i_3}{}_{j_2 k_2} W^{k_2 i_4}{}_{j_3 k_3} ... W^{k_{n-2} i_n}{}_{j_{n-1} j_n} \theta^{j_1} \otimes ... \otimes \theta^{j_n} \quad (2.36)$$

where the matrix W is defined by:

$$\theta^i \wedge \theta^j \equiv W^{ij}{}_{kl} \theta^k \otimes \theta^l = \theta^i \otimes \theta^j - \Lambda^{ij}{}_{kl} \theta^k \otimes \theta^l. \quad (2.37)$$

and $\Lambda^{ij}{}_{kl}$ is the braiding matrix defined by (2.30). The space of n-forms $\Gamma^{\wedge n}$ is therefore defined as in the usual case but with the new permutation operator Λ, and can be shown to be a bicovariant bimodule, with left and right action defined as for $\Gamma \otimes ... \otimes \Gamma$ with the tensor product replaced by the wedge product.

Exterior derivative

Having the exterior product we can define the *exterior derivative*

$$d \ : \ \Gamma \to \Gamma \wedge \Gamma \quad (2.38)$$

$$d(a_k db_k) = da_k \wedge db_k, \quad (2.39)$$

which can easily be extended to $\Gamma^{\wedge n}$ ($d : \Gamma^{\wedge n} \to \Gamma^{\wedge (n+1)}$), and has the following properties:

$$d(\rho \wedge \rho') = d\rho \wedge \rho' + (-1)^k \rho \wedge d\rho' \quad (2.40)$$
$$d(d\rho) = 0 \quad (2.41)$$
$$\mathcal{L}_g(d\rho) = d\mathcal{L}_g \rho \quad (2.42)$$
$$\mathcal{R}_g(d\rho) = d\mathcal{R}_g \rho \quad (2.43)$$

where $\rho \in \Gamma^{\wedge k}$, $\rho' \in \Gamma^{\wedge n}$. The last two properties express the fact that d commutes with the left and right action of G.

Tangent vectors

Using (2.25) to expand df on the basis of the left-invariant one-forms θ^g defines the (left-invariant) tangent vectors t_g:

$$df = \sum_{g \in G} f_g dx^g = \sum_{h \in G'} (\mathcal{R}_{h^{-1}} f - f)\theta^h \equiv \equiv \sum_{h \in G'} (t_h f)\theta^h \quad (2.44)$$

so that the "flat" partial derivatives $t_h f$ are given by

$$t_h f = \mathcal{R}_{h^{-1}} f - f \quad (2.45)$$

The Leibniz rule for the flat partial derivatives t_g reads:

$$t_g(ff') = (t_g f)\mathcal{R}_{g^{-1}} f' + f t_g f' \quad (2.46)$$

In analogy with ordinary differential calculus, the operators t_g appearing in (2.44) are called (left-invariant) *tangent vectors*, and in our case are given by

$$t_g = \mathcal{R}_{g^{-1}} - id \quad (2.47)$$

They satisfy the composition rule:

$$t_g t_{g'} = \sum_h C^h{}_{g,g'} t_h \quad (2.48)$$

where the structure constants are:

$$C^h{}_{g,g'} = \delta^h_{g'g} - \delta^h_g - \delta^h_{g'} \quad (2.49)$$

and have the property:

$$C^{ad(h)g_1}{}_{ad(h)g_2, ad(h)g_3} = C^{g_1}{}_{g_2,g_3} \quad (2.50)$$

Note 2.1 : The exterior derivative on any $f \in Fun(G)$ can be expressed as a commutator of f with the bi-invariant one-form Θ:

$$df = [\Theta, f] \tag{2.51}$$

as one proves by using (2.27) and (2.44).

Note 2.2 : From the fusion rules (2.48) we deduce the "deformed Lie algebra" (cf. ref.s [8, 9, 11]):

$$t_{g_1} t_{g_2} - \Lambda^{g_3,g_4}{}_{g_1,g_2} t_{g_3} t_{g_4} = \boldsymbol{C}^h{}_{g_1,g_2} t_h \tag{2.52}$$

where the $\boldsymbol{C}$ structure constants are given by:

$$\begin{aligned} \boldsymbol{C}^g{}_{g_1,g_2} &\equiv C^g{}_{g_1,g_2} - \Lambda^{g_3,g_4}{}_{g_1,g_2} C^g{}_{g_3,g_4} = \\ &= C^g{}_{g_1,g_2} - C^g{}_{g_2,g_2 g_1 g_2^{-1}} = \\ &= \delta^{ad(g_2^{-1})g}_{g_1} - \delta^g_{g_1} \end{aligned} \tag{2.53}$$

and besides property (2.50) they also satisfy:

$$\boldsymbol{C}^g{}_{g_1,g_2} = \boldsymbol{C}^{g_1}{}_{g,g_2^{-1}} \tag{2.54}$$

Moreover the following identities hold:

i) *deformed Jacobi identities:*

$$\begin{aligned} &\boldsymbol{C}^k{}_{h_1,g_1} \boldsymbol{C}^{h_2}{}_{k,g_2} - \Lambda^{g_3,g_4}{}_{g_1,g_2} \boldsymbol{C}^k{}_{h_1,g_3} \boldsymbol{C}^{h_2}{}_{k,g_4} = \\ &= \boldsymbol{C}^k{}_{g_1,g_2} \boldsymbol{C}^{h_2}{}_{h_1,k} \end{aligned} \tag{2.55}$$

ii) *fusion identities:*

$$\boldsymbol{C}^k{}_{h_1,g} \boldsymbol{C}^{h_2}{}_{k,g'} = C^h{}_{g,g'} \boldsymbol{C}^{h_2}{}_{h_1,h} \tag{2.56}$$

Thus the $\boldsymbol{C}$ structure constants are a representation (the adjoint representation) of the tangent vectors t.

Cartan-Maurer equations, connection and curvature

From the definition (2.24) and eq. (2.27) we deduce the Cartan-Maurer equations:

$$d\theta^g + \sum_{g_1,g_2} C^g{}_{g_1,g_2} \theta^{g_1} \wedge \theta^{g_2} = 0 \tag{2.57}$$

where the structure constants $C^g{}_{g_1,g_2}$ are those given in (2.49).

Parallel transport of the vielbein θ^g can be defined as in ordinary Lie group manifolds:

$$\nabla \theta^g = -\omega^g{}_{g'} \otimes \theta^{g'} \tag{2.58}$$

where $\omega^{g_1}{}_{g_2}$ is the connection one-form:

$$\omega^{g_1}{}_{g_2} = \Gamma^{g_1}{}_{g_3,g_2} \theta^{g_3} \tag{2.59}$$

Thus parallel transport is a map from Γ to $\Gamma \otimes \Gamma$; by definition it must satisfy:

$$\nabla(a\rho) = (da) \otimes \rho + a \nabla \rho, \quad \forall a \in A,\ \rho \in \Gamma \tag{2.60}$$

and it is a simple matter to verify that this relation is satisfied with the usual parallel transport of Riemannian manifolds. As for the exterior differential, ∇ can be extended to a map $\nabla : \Gamma^{\wedge n} \otimes \Gamma \longrightarrow \Gamma^{\wedge (n+1)} \otimes \Gamma$ by defining:

$$\nabla(\varphi \otimes \rho) = d\varphi \otimes \rho + (-1)^n \varphi \nabla \rho \tag{2.61}$$

Requiring parallel transport to commute with the left and right action of G means:

$$\mathcal{L}_h(\nabla \theta^g) = \nabla(\mathcal{L}_h \theta^g) = \nabla \theta^g \tag{2.62}$$

$$\mathcal{R}_h(\nabla \theta^g) = \nabla(\mathcal{R}_h \theta^g) = \nabla \theta^{ad(h)g} \tag{2.63}$$

Recalling that $\mathcal{L}_h(a\rho) = (\mathcal{L}_h a)(\mathcal{L}_h \rho)$ and $\mathcal{L}_h(\rho \otimes \rho') = (\mathcal{L}_h \rho) \otimes (\mathcal{L}_h \rho')$, $\forall a \in A,\ \rho,\ \rho' \in \Gamma$ (and similar for $\mathcal{R}_h$), and substituting (2.58) yields respectively:

$$\Gamma^{g_1}{}_{g_3,g_2} \in \boldsymbol{C} \tag{2.64}$$

and

$$\Gamma^{ad(h)g_1}{}_{ad(h)g_3,ad(h)g_2} = \Gamma^{g_1}{}_{g_3,g_2} \tag{2.65}$$

Therefore the same situation arises as in the case of Lie groups, for which parallel transport on the group manifold commutes with left and right action iff the connection components are $ad(G)$ - conserved constant tensors. As for Lie groups, condition (2.65) is satisfied if one takes Γ proportional to the structure constants. In our case, we can take any combination of the C or $\boldsymbol{C}$ structure constants, since both are $ad(G)$ conserved constant tensors. As we see below, the C constants can be used to define a torsionless connection, while the $\boldsymbol{C}$ constants define a parallelizing connection.

As usual, the *curvature* arises from ∇^2:

$$\nabla^2 \theta^g = -R^g{}_{g'} \otimes \theta^{g'} \tag{2.66}$$

$$R^{g_1}{}_{g_2} \equiv d\omega^{g_1}{}_{g_2} + \omega^{g_1}{}_{g_3} \wedge \omega^{g_3}{}_{g_2} \tag{2.67}$$

The *torsion* R^g is defined by:

$$R^{g_1} \equiv d\theta^{g_1} + \omega^{g_1}{}_{g_2} \wedge \theta^{g_2} \tag{2.68}$$

Using the expression of ω in terms of Γ and the Cartan-Maurer equations yields

$$\begin{aligned} &R^{g_1}{}_{g_2} = \\ &= (-\Gamma^{g_1}{}_{h,g_2} C^h{}_{g_3,g_4} + \Gamma^{g_1}{}_{g_3,h}\Gamma^h{}_{g_4,g_2})\ \theta^{g_3} \wedge \theta^{g_4} = \\ &= (-\Gamma^{g_1}{}_{h,g_2} \boldsymbol{C}^h{}_{g_3,g_4} + \Gamma^{g_1}{}_{g_3,h}\Gamma^h{}_{g_4,g_2} - \\ &\quad \Gamma^{g_1}{}_{g_4,h}\Gamma^h{}_{g_4 g_3 g_4^{-1},g_2})\ \theta^{g_3} \otimes \theta^{g_4} \end{aligned}$$

$$\begin{aligned} &R^{g_1} = (-C^{g_1}{}_{g_2,g_3} + \Gamma^{g_1}{}_{g_2,g_3})\ \theta^{g_2} \wedge \theta^{g_3} = \\ &(-\boldsymbol{C}^{g_1}{}_{g_2,g_3} + \Gamma^{g_1}{}_{g_2,g_3} - \Gamma^{g_1}{}_{g_3,g_3 g_2 g_3^{-1}}) \\ &\theta^{g_2} \otimes \theta^{g_3} \end{aligned} \tag{2.69}$$

Thus a connection satisfying:

$$\Gamma^{g_1}{}_{g_2,g_3} - \Gamma^{g_1}{}_{g_3,g_3 g_2 g_3^{-1}} = \boldsymbol{C}^{g_1}{}_{g_2,g_3} \tag{2.70}$$

corresponds to a vanishing torsion $R^g = 0$ and could be referred to as a "Riemannian" connection.

On the other hand, the choice:

$$\Gamma^{g_1}{}_{g_2,g_3} = \boldsymbol{C}^{g_1}{}_{g_3,g_2^{-1}} \tag{2.71}$$

corresponds to a vanishing curvature $R^g{}_{g'} = 0$, as can be checked by using the fusion equations (2.56) and property (2.54). Then (2.71) can be called the parallelizing connection: *finite groups are parallelizable.*

Tensor transformations

Under the familiar transformation of the connection 1-form:

$$(\omega^i{}_j)' = a^i{}_k \omega^k{}_l (a^{-1})^l{}_j + a^i{}_k d(a^{-1})^k{}_j \tag{2.72}$$

the curvature 2-form transforms homogeneously:

$$(R^i{}_j)' = a^i{}_k R^k{}_l (a^{-1})^l{}_j \tag{2.73}$$

The transformation rule (2.72) can be seen as induced by the change of basis $\theta^i = a^i{}_j \theta^j$, with $a^i{}_j$ invertible x-dependent matrix (use eq. (2.60) with $a\rho = a^i{}_j \theta^j$).

Metric

The metric tensor g can be defined as an element of $\Gamma \otimes \Gamma$:

$$g = g_{i,j} \theta^i \otimes \theta^j \tag{2.74}$$

Requiring it to be invariant under left and right action of G means:

$$\mathcal{L}_h(g) = g = \mathcal{R}_h(g) \tag{2.75}$$

or equivalently, by recalling $\mathcal{L}_h(\theta^i \otimes \theta^j) = \theta^i \otimes \theta^j$, $\mathcal{R}_h(\theta^i \otimes \theta^j) = \theta^{ad(h)i} \otimes \theta^{ad(h)j}$:

$$g_{i,j} \in \boldsymbol{C}, \quad g_{ad(h)i,ad(h)j} = g_{i,j} \tag{2.76}$$

These properties are analogous to the ones satisfied by the Killing metric of Lie groups, which is indeed constant and invariant under the adjoint action of the Lie group.

On finite G there are various choices of biinvariant metrics. One can simply take $g_{i,j} = \delta_{i,j}$, or $g_{i,j} = \boldsymbol{C}^k{}_{l,i} \boldsymbol{C}^l{}_{k,j}$.

For any biinvariant metric g_{ij} there are tensor transformations $a^i{}_j$ under which g_{ij} is invariant, i.e.:

$$a^h{}_{h'} \gamma_{h,k} a^k{}_{k'} = \gamma_{h'k'} \Leftrightarrow a^h{}_{h'} \gamma_{h,k} = \gamma_{h'k'} (a^{-1})^{k'}_k \tag{2.77}$$

These transformations are simply given by the matrices that rotate the indices according to the adjoint action of G:

$$a^h{}_{h'}(g) = \delta^{ad(\alpha(g))h}_{h'} \tag{2.78}$$

where $\alpha(g) : G \mapsto G$ is an arbitrary mapping. Then these matrices are functions of G via this mapping, and their action leaves γ invariant because of the its biinvariance (2.76). Indeed substituting these matrices in (2.77) yields:

$$\begin{aligned} &a^h{}_{h'}(g) \gamma_{h,k} a^k{}_{k'}(g) = \\ &\gamma_{ad([\alpha(g)]^{-1})h',ad([\alpha(g)]^{-1})k'} = \gamma_{h',k'} \end{aligned} \tag{2.79}$$

proving the invariance of γ.

Consider now a contravariant vector φ^i transforming as $(\varphi^i)' = a^i{}_j(\varphi^j)$. Then using (2.77) one can easily see that

$$(\varphi^k \gamma_{k,i})' = \varphi^{k'} \gamma_{k',i'} (a^{-1})^{i'}_i \tag{2.80}$$

i.e. the vector $\varphi_i \equiv \varphi^k \gamma_{k,i}$ indeed transforms as a covariant vector.

Lie derivative and diffeomorphisms

The notion of diffeomorphisms, or general coordinate transformations, is fundamental in gravity theories. Is there such a notion in the setting of differential calculi on Hopf algebras ? The answer is affirmative, and has been discussed in detail in ref.s [9, 10, 11]. As for differentiable manifolds, it relies on the existence of the Lie derivative.

Let us review the situation for Lie group manifolds. The Lie derivative l_{t_i} along a left-invariant tangent vector t_i is related to the infinitesimal right translations generated by t_i:

$$l_{t_i}\rho = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\mathcal{R}_{\exp[\varepsilon t_i]}\rho - \rho] \tag{2.81}$$

ρ being an arbitrary tensor field. Introducing the coordinate dependence

$$l_{t_i}\rho(y) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[\rho(y + \varepsilon t_i) - \rho(y)] \tag{2.82}$$

identifies the Lie derivative l_{t_i} as a directional derivative along t_i. Note the difference in meaning of the symbol t_i in the r.h.s. of these two equations: a group generator in the first, and the corresponding tangent vector in the second.

For finite groups the Lie derivative takes the form:

$$l_{t_g}\rho = [\mathcal{R}_{g^{-1}}\rho - \rho] \tag{2.83}$$

so that the Lie derivative is simply given by

$$l_{t_g} = \mathcal{R}_{g^{-1}} - id = t_g \tag{2.84}$$

cf. the definition of t_g in (2.47). For example

$$l_{t_g}(\theta^{g_1} \otimes \theta^{g_2}) = \theta^{ad(g^{-1})g_1} \otimes \theta^{ad(g^{-1})g_2} - \theta^{g_1} \otimes \theta^{g_2} \tag{2.85}$$

As in the case of differentiable manifolds, the Cartan formula for the Lie derivative acting on p-forms holds:

$$l_{t_g} = i_{t_g} d + d i_{t_g} \tag{2.86}$$

see ref.s [9, 11, 7].

Exploiting this formula, diffeomorphisms (Lie derivatives) along generic tangent vectors V can also be consistently defined via the operator:

$$l_V = i_V d + d i_V \tag{2.87}$$

This requires a suitable definition of the contraction operator i_V along generic tangent vectors V, discussed in ref. [11, 7].

We have then a way of defining "diffeomorphisms" along arbitrary (and x-dependent) tangent vectors for any tensor ρ:

$$\delta\rho = l_V \rho \tag{2.88}$$

and of testing the invariance of candidate lagrangians under the generalized Lie derivative.

Haar measure and integration

Since we want to be able to define actions (integrals on p-forms) we must now define integration of p-forms on finite groups.

Let us start with integration of functions f. We define the integral map h as a linear functional $h : Fun(G) \mapsto \boldsymbol{C}$ satisfying the left and right invariance conditions:

$$h(\mathcal{L}_g f) = 0 = h(\mathcal{R}_g f) \tag{2.89}$$

Then this map is uniquely determined (up to a normalization constant), and is simply given by the "sum over G" rule:

$$h(f) = \sum_{g \in G} f(g) \tag{2.90}$$

Next we turn to define the integral of a p-form. Within the differential calculus we have a basis of left-invariant 1-forms, which may allow the definition of a biinvariant volume element. In general for a differential calculus with n independent tangent vectors, there is an integer $p \geq n$ such that the linear space of p-forms is 1-dimensional, and $(p+1)$- forms vanish identically. We will see explicit examples in the next Section. This means that every product of p basis one-forms $\theta^{g_1} \wedge \theta^{g_2} \wedge ... \wedge \theta^{g_p}$ is proportional to one of these products, that can be chosen to define the volume form vol:

$$\theta^{g_1} \wedge \theta^{g_2} \wedge ... \wedge \theta^{g_p} = \epsilon^{g_1, g_2, \cdots g_p} vol \tag{2.91}$$

where $\epsilon^{g_1,g_2,\cdots g_p}$ is the proportionality constant. Note that the volume p-form is obviously left invariant. We can prove that it is also right invariant with the following argument. Suppose that vol be given by $\theta^{h_1} \wedge \theta^{h_2} \wedge ... \wedge \theta^{h_p}$ where $h_1, h_2, ...h_p$ are given group element labels. Then the right action on vol yields:

$$\mathcal{R}_g[\theta^{h_1} \wedge ... \wedge \theta^{h_p}] = \theta^{ad(g)h_1} \wedge ... \wedge \theta^{ad(g)h_p} = \epsilon^{ad(g)h_1,...ad(g)h_p} vol \tag{2.92}$$

Recall now that the "epsilon tensor" ϵ is necessarily made out of the W tensors of eq. (2.37), defining the wedge product. These tensors are invariant under the adjoint action $ad(g)$, and so is the ϵ tensor. Therefore $\epsilon^{ad(g)h_1,...ad(g)h_p} = \epsilon^{h_1,...h_p} = 1$ and $\mathcal{R}_g vol = vol$. This will be verified in the examples of next Section.

Having identified the volume p-form it is natural to set

$$\int f vol \equiv h(f) = \sum_{g \in G} f(g) \tag{2.93}$$

and define the integral on a p-form ρ as:

$$\int \rho = \int \rho_{g_1,...g_p}\ \theta^{g_1} \wedge ... \wedge \theta^{g_p} = \int \rho_{g_1,...g_p}\ \epsilon^{g_1,...g_p} vol \equiv \equiv \sum_{g \in G} \rho_{g_1,...g_p}(g)\ \epsilon^{g_1,...g_p} \tag{2.94}$$

Due to the biinvariance of the volume form, the integral map $\int : \Gamma^{\wedge p} \mapsto \boldsymbol{C}$ satisfies the biinvariance conditions:

$$\int \mathcal{L}_g f = \int f = \int \mathcal{R}_g f \tag{2.95}$$

Moreover, under the assumption that the volume form belongs to a nontrivial cohomology class, that is $d(vol) = 0$ but $vol \neq d\rho$, the important property holds:

$$\int df = 0 \tag{2.96}$$

with f any $(p-1)$-form: $f = f_{g_2,...g_p}\ \theta^{g_2} \wedge ... \wedge \theta^{g_p}$. This property, which allows integration by parts, has a simple proof. Rewrite $\int df$ as:

$$\int df = \int (df_{g_2,...g_p}) \theta^{g_2} \wedge ... \wedge \theta^{g_p} + + \int f_{g_2,...g_p} d(\theta^{g_2} \wedge ... \wedge \theta^{g_p}) \tag{2.97}$$

Under the cohomology assumption the second term in the r.h.s. vanishes, since $d(\theta^{g_2} \wedge ... \wedge \theta^{g_p}) = 0$ (otherwise, being a p-form, it should be proportional to vol, and this would contradict the assumption $vol \neq d\rho$). Using now (2.44) and (2.93):

$$\int df = \int (t_{g_1} f_{g_2,...g_p}) \theta^{g_1} \wedge \theta^{g_2} \wedge ... \wedge \theta^{g_p} = \int [\mathcal{R}_{g_1^{-1}} f_{g_2,...g_p} - f_{g_2,...g_p}] \epsilon^{g_1,...g_p} vol = \epsilon^{g_1,...g_p} \sum_{g \in G} [\mathcal{R}_{g_1^{-1}} f_{g_2,...g_p}(g) - f_{g_2,...g_p}(g)] = = 0 \tag{2.98}$$

Q.E.D.

3. Bicovariant calculus on S_3

In this Section we illustrate the general theory on the particular example of the permutation group S_3.

Elements: $a = (12)$, $b = (23)$, $c = (13)$, $ab = (132)$, $ba = (123)$, e.

Nontrivial conjugation classes: $I = [a, b, c]$, $II = [ab, ba]$.

There are 3 bicovariant calculi BC_I, BC_{II}, BC_{I+II} corresponding to the possible unions of the conjugation classes [5]. They have respectively dimension 3, 2 and 5. We examine here the BC_I and BC_{II} calculi.

BC_I differential calculus

Basis of the 3-dimensional vector space of one-forms:

$$\theta^a,\ \theta^b,\ \theta^c \tag{3.1}$$

Basis of the 4-dimensional vector space of two-forms:

$$\theta^a \wedge \theta^b,\ \theta^b \wedge \theta^c,\ \theta^a \wedge \theta^c,\ \theta^c \wedge \theta^b \tag{3.2}$$

Every wedge product of two θ can be expressed as linear combination of the basis elements:

$$\begin{aligned} \theta^b \wedge \theta^a &= -\theta^a \wedge \theta^c - \theta^c \wedge \theta^b, \\ \theta^c \wedge \theta^a &= -\theta^a \wedge \theta^b - \theta^b \wedge \theta^c \end{aligned} \tag{3.3}$$

Basis of the 3-dimensional vector space of three-forms:

$$\theta^a \wedge \theta^b \wedge \theta^c,\ \theta^a \wedge \theta^c \wedge \theta^b,\ \theta^b \wedge \theta^a \wedge \theta^c \quad (3.4)$$

and we have:

$$\begin{aligned}
&\theta^c \wedge \theta^b \wedge \theta^a = -\theta^c \wedge \theta^a \wedge \theta^c = \\
&\quad -\theta^a \wedge \theta^c \wedge \theta^a = \theta^a \wedge \theta^b \wedge \theta^c \\
&\theta^b \wedge \theta^c \wedge \theta^a = -\theta^b \wedge \theta^a \wedge \theta^b = \\
&\quad -\theta^a \wedge \theta^b \wedge \theta^a = \theta^a \wedge \theta^c \wedge \theta^b \\
&\theta^c \wedge \theta^a \wedge \theta^b = -\theta^c \wedge \theta^b \wedge \theta^c = \\
&\quad -\theta^b \wedge \theta^c \wedge \theta^b = \theta^b \wedge \theta^a \wedge \theta^c
\end{aligned} \quad (3.5)$$

Basis of the 1-dimensional vector space of four-forms:

$$vol = \theta^a \wedge \theta^b \wedge \theta^a \wedge \theta^c \quad (3.6)$$

and we have:

$$\theta^{g_1} \wedge \theta^{g_2} \wedge \theta^{g_3} \wedge \theta^{g_4} = \epsilon^{g_1,g_2,g_3,g_4} vol \quad (3.7)$$

where the nonvanishing components of the ϵ tensor are:

$$\begin{aligned}
&\epsilon_{abac} = \epsilon_{acab} = \epsilon_{cbca} = \epsilon_{cacb} = \epsilon_{babc} = \epsilon_{bcba} = 1 \\
&\epsilon_{baca} = \epsilon_{caba} = \epsilon_{abcb} = \epsilon_{cbab} = \epsilon_{acbc} = \epsilon_{bcac} = -1
\end{aligned}$$

Cartan-Maurer equations:

$$\begin{aligned}
&d\theta^a + \theta^b \wedge \theta^c + \theta^c \wedge \theta^b = 0 \\
&d\theta^b + \theta^a \wedge \theta^c + \theta^c \wedge \theta^a = 0 \\
&d\theta^c + \theta^a \wedge \theta^b + \theta^b \wedge \theta^a = 0
\end{aligned} \quad (3.8)$$

The exterior derivative on any three-form of the type $\theta \wedge \theta \wedge \theta$ vanishes, as one can easily check by using the Cartan-Maurer equations and the equalities between exterior products given above. Then, as shown in the previous Section, integration of a total differential vanishes on the "group manifold" of S_3 corresponding to the BC_I bicovariant calculus. This "group manifold" has three independent directions, associated to the cotangent basis $\theta^a,\ \theta^b,\ \theta^c$. Note however that the volume element is of order four in the left-invariant one-forms θ.

BC_{II} differential calculus

Basis of the 2-dimensional vector space of one-forms:

$$\theta^{ab},\ \theta^{ba} \quad (3.9)$$

Basis of the 1-dimensional vector space of two-forms:

$$vol = \theta^{ab} \wedge \theta^{ba} = -\theta^{ba} \wedge \theta^{ab} \quad (3.10)$$

so that:

$$\theta^{g_1} \wedge \theta^{g_2} = \epsilon^{g_1,g_2} vol \quad (3.11)$$

where the ϵ tensor is the usual 2-dimensional Levi-Civita tensor.

Cartan-Maurer equations:

$$d\theta^{ab} = 0, \quad d\theta^{ba} = 0 \quad (3.12)$$

Thus the exterior derivative on any one-form θ^g vanishes and integration of a total differential vanishes on the group manifold of S_3 corresponding to the BC_{II} bicovariant calculus. This group manifold has two independent directions, associated to the cotangent basis $\theta^{ab},\ \theta^{ba}$.

Visualization of the S_3 group "manifold"

We can draw a picture of the group manifold of S_3. It is made out of 6 points, corresponding to the group elements and identified with the functions $x^e, x^a, x^b, x^c, x^{ab}, x^{ba}$.

BC_I - calculus:

From each of the six points x^g one can move in three directions, associated to the tangent vectors t_a, t_b, t_c, reaching three other points whose "coordinates" are

$$\mathcal{R}_a x^g = x^{ga},\ \ \mathcal{R}_b x^g = x^{gb},\ \ \mathcal{R}_c x^g = x^{gc} \quad (3.13)$$

The 6 points and the "moves" along the 3 directions are illustrated in the Fig. 1. The links are not oriented since the three group elements a, b, c coincide with their inverses.

BC_{II} - calculus:

From each of the six points x^g one can move in two directions, associated to the tangent vectors t_{ab}, t_{ba}, reaching two other points whose "coordinates" are

$$\mathcal{R}_{ab}x^g = x^{gba}, \quad \mathcal{R}_{ba}x^g = x^{gab} \tag{3.14}$$

The 6 points and the "moves" along the 3 directions are illustrated in Fig. 1. The arrow convention on a link labeled (in italic) by a group element h is as follows: one moves in the direction of the arrow via the action of $\mathcal{R}_h$ on x^g. (In this case $h = ab$). To move in the opposite direction just take the inverse of h.

The pictures in Fig. 1 characterize the bicovariant calculi BC_I and BC_{II} on S_3, and were drawn in ref. [5] as examples of digraphs, used to characterize different calculi on sets. Here we emphasize their geometrical meaning as finite group "manifolds".

Acknowledgments

It is a pleasure to thank the organizers of the Corfu Summer Institute on Elementary Particle Physics for their invitation to discuss physics in such a beautiful and relaxed atmosphere.

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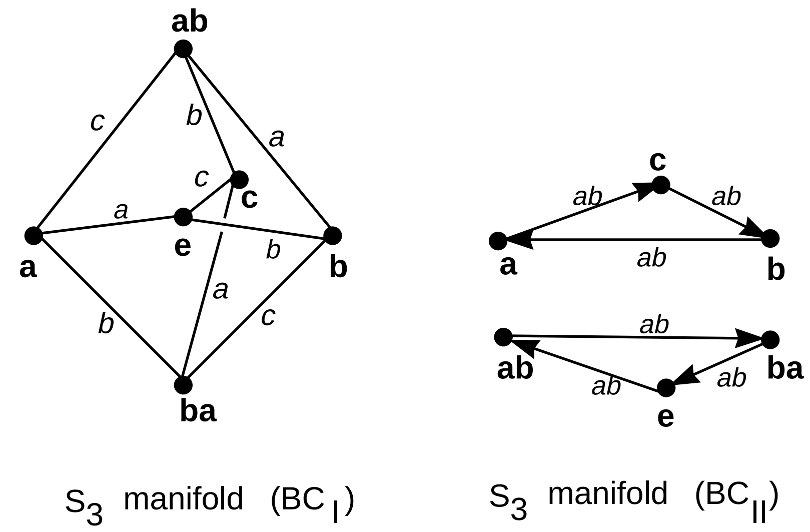

Figure 1: S_3 group manifold, and moves of the points under the group action